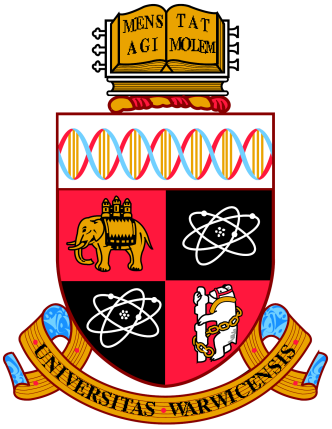

Space-time localisation for the dynamic Φ^4 model

by

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Contents

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Pépé, tu es parti juste après que j'ai soutenu ma thèse, j'espère que je te donne un peu de fierté.

Declarations

Parts of this thesis have been previously published by the author in the following:

[51] Augustin Moinat and Hendrik Weber. Local bounds for stochastic reaction diffusion equations. *arXiv preprint arXiv:1808.10401*, 2018

[52] Augustin Moinat and Hendrik Weber. Space-time localisation for the dynamic Φ^4_3 model. *arXiv e-prints*, art. arXiv:1811.05764, Nov 2018

[21] Ajay Chandra, Augustin Moinat, and Hendrik Weber. A priori bounds for the ϕ^4 equation in the full sub-critical regime. *arXiv preprint arXiv:1910.13854*, 2019

Hendrik Weber and Ajay Chandra, co-authors of all or part of these works, agree that Augustin Moinat took a prominent part in the research and creation of these articles, and agree to the inclusion of those works in the present thesis.

Signed,

Ajay Chandra

Hendrik Weber

Chapter 1

Introduction

The focus of this thesis is the study of nonlinear stochastic partial differential equations (SPDEs) of the type

$$(\partial_t - \Delta)u = -f(u) + \zeta, \tag{1.0.1}$$

over $\mathbb{R}_t \times \mathbb{R}^d_x$ where ζ is an irregular distribution and f is super-linear drift with $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

The particular example we have in mind and that motivated this work is the dynamic Φ^4 equation, where $f(u) = u^3$ and ζ is the space-time white noise. This equation was introduced in constructive quantum field theory as a tool to construct the Φ^4 measure. In the spirit of stochastic quantization (Symanzik [67], Nelson [57, 58], Parisi and Wu[64]), this equation introduces a reversible stochastic dynamic, and the target measure is the invariant measure of this dynamic.

This equation has also been one of the motivating examples of the theory of regularity structures introduced by Hairer in [41]. Regularity structures give a rigorous meaning and a local solution theory for large class of semi-linear SPDEs which require renormalisation due to the low expected regularity of the solution. The theory of regularity structures has been developed into an impressive machinery in [10, 11, 18].

Up to now it remains a local solution theory. That is due to the fact that the theory of regularity structures states a mild formulation of the equation to solve, and disregards large scale properties such as conserved quantities or damping terms. For example, the regularity structure for equation (1.0.1) doesn't depend on the sign $-f(u)$. The present work is part of a program to study long term existence of singular SPDEs, following [54, 55] where similar results were obtained on the torus. We show one way to use other analytic tools to obtain large scale results on the full space $\mathbb{R}^d$, which are necessary to construct an invariant measure.

In Chapter 2 we prove space-time "coming down from infinity" for the solution u when $\zeta \in C^{\alpha-2}$ for $\alpha > 0$, in the sense that we are able to show bounds on the solution u on a compact set $D \subset \mathbb{R}_t \times \mathbb{R}^d_x$ that depend only on the rough driving term on a slightly larger set, in particular independent of initial and boundary conditions. For a smooth driving term this bound is given by a maximum principle which exploits the super-linear negative drift $-f(u)$. We see how to link this large scale result with small scale Schauder theory in this simpler case which does not require renormalisation. This chapter contains the results of [51], written in collaboration with Hendrik Weber and accepted in the *Electronic Journal of Probability*.

In Chapter 3 we prove a similar result for the Φ^4 equation in 3 dimensions. In this relatively simple case of singular SPDE, the regularity structure associated can be handled by hand but all the analytical tools are introduced. This chapter contains the results of [52], written in collaboration with Hendrik Weber and accepted in *Communications on Pure and Applied Mathematics*.

In Chapter 4 we extend this result to more singular noise. We reformulate the theory of regularity structures and show how to derive those large scale results in this context. This chapter contains the results of [21], written in collaboration with Ajay Chandra and Hendrik Weber. It has only just been submitted for publication.

Those articles were all self-contained, but they are a natural continuity of each other. Therefore, the adaptation to make it into a coherent thesis was an easy work. However, they all require, for a full understanding of their implications, some background knowledge in stochastic PDEs, regularity structures and constructive quantum field theory. We will now try to give a short introduction in those domains.

1.1 Φ^4 field theory

1.1.1 The Φ^4 measure

Formally, one can express the Φ^4 measure on $\mathbb{R}^d$ as

$$d\mu(\phi) \propto \exp\Big(- \int_{\mathbb{R}^d} \phi^4 - m^2\phi^2 + \frac{1}{2}|\nabla\phi|^2 \Big) \prod_{x\in\mathbb{R}^d} d\phi(x).$$

In constructive quantum field theory, a set of axioms [60, 61] gives a way to build a field theory from a measure. A lot of effort has consequently been invested in finding non-trivial measures that satisfied these axioms in dimension 4, which is the physical space-time dimension.

The Φ^4 measure has a particular place in constructive quantum field theory as the first

example of measure on $\mathbb{R}^3$ satisfying those axioms. In two dimensions, the existence of the Φ^4 measure was proven early on [32, 65] while in dimensions four and higher, another approach is necessary [26].

In three dimensions, the Φ^4 measure has been constructed in several ways since the original construction by a *phase cell expansion* [25, 31] in the 70's. Among those ways are Balaban's *block average method* [8], *skeleton inequalities method* by Brydges, Fröhlich and Sokal [13], both from 1983, and the *renormalisation group* method of Kupiainen in 2016 [50]. The progress in singular stochastic PDEs in the past 5 years gives a new approach to the problem via stochastic quantization, introduced in Section 1.1.2.

An approximation procedure can be used to build this measure, starting with a measure on a periodic lattice $\Lambda_{M,\epsilon} = (\epsilon(\mathbb{Z}/M\mathbb{Z}))^3$, and then letting $\epsilon \to 0$ and $M \to \infty$.

$$d\mu_{M,\epsilon}(\phi) \propto \exp\Big(- \sum_{\Lambda_{M,\epsilon}} \phi^4 - (m^2 + C_{M,\epsilon})\phi^2 + C'_{M,\epsilon} + \frac{1}{2}|\nabla_\epsilon \phi|^2 \Big) \prod_{x \in \Lambda_{M,\epsilon}} d\phi(x).$$

However in dimensions 2 and 3, convergence of the measure in the limit $\epsilon \to 0$, $M \to \infty$ requires a fine tuning of the *renormalisation* parameters $C_{M,\epsilon}$ and $C'_{M,\epsilon}$ which have to diverge as ϵ vanishes. The divergence rate of these constants depends on the dimension. In dimension 2 we have $C_{M,\epsilon} \propto -\log(\epsilon)$ while in dimension 3, $C_{M,\epsilon} \propto \epsilon^{-1}$.

The Φ^4 measure also arises as a scaling limit for the Ising model with local interaction. In two dimensions, the Ising model is a spin model with values $\sigma \in \{1, -1\}$ on the lattice $\Lambda_N = \mathbb{Z}^2/(2N+1)\mathbb{Z}^2$. The interaction is described by the Hamiltonian

$$H_\gamma(\sigma) = -\frac{1}{2} \sum_{k,j \in \Lambda_N} h_\gamma(k-j)\sigma_i\sigma_j,$$

where h_γ is an interaction with range $\gamma^{-1} \sim \sqrt{N}$. The Gibbs measure λ_N for inverse temperature β is defined on $\{\pm 1\}^{\Lambda_N}$ as

$$\lambda_N(\sigma) = Z_N^{-1} \exp(-\beta H_\gamma(\sigma)),$$

where

$$Z_N = \sum_{\sigma \in \{\pm 1\}^{\Lambda_N}} \exp(-\beta H_\gamma(\sigma)).$$

Under suitable rescaling the invariant measure for this model converges to the Φ^4 measure. In [53], the authors show a dynamical version of this result. The Glauber dynamics associated with this measure are described by the rate at which a spin at site

$j \in \Lambda_N$ can flip:

$$c_N(\sigma, j) = \frac{\lambda_N(\sigma^j)}{\lambda_N(\sigma) + \lambda_N(\sigma^j)},$$

where σ^j is the configuration σ with the spin site j reversed. This model converges under suitable rescaling to solutions to the dynamic Φ^4 model, which we introduce in the next section.

1.1.2 Stochastic quantization

The idea of stochastic quantization is derived from the Monte-Carlo method for Markov chains. A measure prescribed by a Hamiltonian is described as the invariant measure of a dynamical system with state space the space of the measure, and transition probability given by the flow of the Hamiltonian.

Following that idea, the dynamic Φ^4 equation was introduced to get the Φ^4 measure as an equilibrium limit, here with mass 0:

$$(\partial_t - \Delta)u = -u^3 + \zeta, \tag{1.1.1}$$

where ζ is the space-time white noise in $d+1$ dimensions. Constructing solutions to this equation in order to construct the measure has been suggested since 1964 in works by Nelson [57, 58], Parisi and Wu [64], and Symanzik [67], but the solution theory for these equations remained i ncomplete until recently. One way to prove existence of equilibrium solutions is to prove that solutions converge to a compact set, and then prove contractivity within this set. This is the approach taken in [70] for the Φ^4 equation on the 2-dimensional torus.

During the same period as the research for this thesis was done, the Φ^4 measure was constructed in dimension 3 using stochastic quantization for the first time by Gubinelli and Hofmanovà [36], using bounds in weighted Besov spaces. Our results differ from theirs and from other a priori estimates including "coming-down from infinity" properties that have been proven for singular SPDEs, namely the dynamic ϕ^{2m}_2 [55, 68] and ϕ^4_3 models [2, 54] both on compact domains and on the full space. These works all relied on Fourier methods, the method of paracontrolled distributions, rather than the theory of regularity structures. Some of the bounds obtained there imply coming down from infinity in time only, in the case of ϕ^4_3 on the full space [35] in a weighted space. The ideas presented here extend to more singular cases within the theory of regularity structures. This is the content of Chapters 3 and 4. There we show that our method significantly simplifes the technical arguments used in [2, 35, 54] and extend its scope to construct solutions on the full space without the need for weights.

1.1.3 The need for renormalisation

The Φ^4 equation features the nonlinearity $-u^3$ and space-time white noise over $\mathbb{R}^{d+1}$, which is a family $\{\zeta(h),\ h \in L^2(\mathbb{R}^{d+1})\}$ of centred Gaussian random variables such that

$$\mathbb{E}[\zeta(h)^2] = \|h\|_{L^2} \tag{1.1.2}$$

Formally, one can think of it as a space-time process where $\mathbb{E}[\zeta(z)\zeta(z')] = \delta_{z,z'}$. The existence of such object follows from the Kolmogorov extension theorem [55, Lemma 9] but a constructive approach can also be formulated, for example on the d-dimensional torus $\mathbb{T}^d$ where the space-time white noise is defined as

$$\zeta(\phi) = \sum_{\omega \in \mathbb{Z}^d} \int_{\mathbb{R}} \widehat{\phi}(t,\omega) dW(t,\omega)$$

where $W = (W(\cdot,\omega))_{\omega\in\mathbb{Z}}$ is a family of complex valued Brownian motion with variance

$$\mathbb{E}[W(t,\omega)W(t,\omega')] = \begin{cases} |t| & \text{if } \omega = -\omega' \\ 0 & \text{otherwise.} \end{cases}$$

Apart from the negative renormalisation, explained in Section 1.2.3, all that we need for the analysis performed in this thesis is the regularity of a sample path. In spatial dimension d, the space-time white noise has regularity $-\frac{d+2}{2} - \epsilon$ for any $\epsilon > 0$, when measured in the parabolic metric suggested by the parabolic equation (1.1.5).

Classical Schauder estimates state that the gain of regularity for the inverse heat operator is 2, which means that at best, u is of regularity $-\frac{d-2}{2} - \epsilon$. In dimension 2 and more, u is a distribution and the nonlinearity $-u^3$ is not classically defined. We then call the Φ^4 equation a *singular* SPDE . On the other hand, assuming we can give a meaning to multiplication of distributions, their regularities would add up and the nonlinearity $-u^3$ has regularity at best $3(-\frac{d-2}{2} - \epsilon)$. Therefore, in dimension 4 and greater, it is of lower regularity than the noise. We do not expect to have a solution in that *supercritical* case, which is also the dimension in which other constructions of the measure break down. In dimension between 2 and 4, the nonlinearity is still classically ill-defined but there are several ways to locally define a solution to the *subcritical* singular SPDE.

The criticality can also be seen from the following scaling argument. The stochastic heat equation $(\partial_t - \Delta)u = \zeta$ is invariant in law under the scaling

$$\widehat{u}(t,x) = \lambda^{\frac{d}{2}-1} u(\lambda^2 t, \lambda x). \tag{1.1.3}$$

In the Φ^4 equation, the nonlinearity $-u^3$ scales like $-\lambda^{4-d}\widehat{u}^3$ under this scaling. For

$d < 4$, the nonlinearity formally vanishes on small scales which suggests that the solution can be locally described by the solution to the linear heat equation.

This was first done in dimension 2 in [22] by expanding around the solution to the linear equation. In practice, one defines a smooth version of the noise, either convolving with a smooth, compactly supported approximation of unity $\Psi_\delta(x,t) = \frac{1}{\delta^{d+2}}\Psi(\frac{t}{\delta^2}, \frac{x}{\delta})$ and setting $\zeta_\delta = \zeta * \Psi_\delta$ or, on the d-dimensional torus, by performing a truncation in Fourier modes

$$\zeta_\delta(\phi) = \sum_{|\omega|<\delta^{-1}} \int_{\mathbb{R}} \widehat{\phi}(t,\omega) dW(t,\omega). \tag{1.1.4}$$

The following smooth equation can be solved classically, but the term u_δ^3 does not converge in the limit $\delta \to 0$.

$$(\partial_t - \Delta) u_\delta = -u_\delta^3 + \zeta_\delta. \tag{1.1.5}$$

Therefore we define ϖ_δ be a solution to

$$(\partial_t - \Delta)\varpi_\delta = \zeta_\delta.$$

Then $v_\delta = u_\delta - \varpi_\delta$ is solution to

$$(\partial_t - \Delta) v_\delta = -v_\delta^3 - 3v_\delta^2 \varpi_\delta - 3v_\delta \varpi_\delta^2 - \varpi_\delta^3. \tag{1.1.6}$$

However, the products $v_\delta^2\varpi_\delta$, $v_\delta\varpi_\delta^2$ and ϖ_δ^3 are still ill-defined: they are not expected to converge as δ vanishes. It was expected from the study of the Φ^4 measure that some renormalisation would have to be introduced. It takes here the form of some "infinite counter-terms" that are subtracted from the nonlinearity.

$$(\partial_t - \Delta) u_\delta = -u_\delta^3 + 3C_\delta u_\delta + \zeta_\delta, \tag{1.1.7}$$

where the constant C_δ diverges as δ goes to 0. We discuss this in more details in Section 1.2.3. If u_δ solves equation (1.1.7), then v_δ solves the remainder equation which we rearrange as follows

$$(\partial_t - \Delta) v_\delta = -v_\delta^3 - 3v_\delta^2 \varpi - 3v_\delta(\varpi^2 - C_\delta) - (\varpi^3 - 3C_\delta \varpi). \tag{1.1.8}$$

Provided one can prove that $(\varpi_\delta^3 - 3C_\delta\varpi)$ converges to a distribution $\varv \in C^{-3\epsilon}$ as δ vanishes, and $(\varpi_\delta^2 - C_\delta)$ to $\varv \in C^{-2\epsilon}$, then v should converge to a $C^{2-3\epsilon}$ function, at which point the products $v^2\varpi$ and $v\varv$ are well defined.

The terms $\varv$ and $\varv$ are called renormalised products. In practice, one defines all

renormalised products and then has to show how they modify the equation. This is done in great generality with the machinery of regularity structures, which we introduce in the next section.

1.2 Regularity Structures

Before we introduce the theory of regularity structures as in [41], let us remind the reader that Chapters 2 and 3 of this thesis do not need the machinery of regularity structures, the first one being set in the non-singular case, and in the second one all the computations are done by hand. In Chapter 4, our approach is in some sense closer to the theory of rough paths from [34], which was also an inspiration for the theory of regularity structures.

In both rough-path theory and regularity structures, a non-continuous solution map associating a distribution $u_\delta \in \mathcal{D}'(\mathbb{R}^d)$ to (a smooth version of) a realisation of the noise $\zeta_\delta \in \mathcal{D}'(\mathbb{R}^d)$ is split into two parts: a lift of the noise to a richer space, either a rough path above the noise or the space of *models*, and then a continuous solution map from the space of rough path or models to distributions, where the solution u_δ lives.

Elements of the rough path and the model and relations between them are described with some tensor algebras in the case of rough path, and with an abstract space T with a structure group G in the case of regularity structures. In Chapter 4 we do build an abstract space but only essential elements of the structure group G are introduced.

In the case of regularity structures however, the "canonical" lift, which gives the multiplicative model corresponding to equation (1.1.5) does not converge in the space of models. A renormalised model is introduced, corresponding to equation (1.1.7), and the stochastic estimates presented in Section 1.2.3 ensure that this model converges.

We will keep the Φ^4 equation as an example, but regularity structures in full generality are defined for equations or sets of equations of the type

$$\mathcal{L}u = P(u) + \zeta,$$

where $\mathcal{L}$ is a differential operator and P is a nonlinear operator in u and derivatives of u of order strictly less than the order of $\mathcal{L}$. The term ζ is a noise term typically of low regularity, which makes the nonlinearity P ill-defined. Some classical examples are the following:

- The KPZ equation describing the growth of an interface :

$$(\partial_t - \partial_x^2)h = (\partial_x h)^2 + \zeta.$$

- The parabolic Anderson model in dimensions 2 and 3 :

$$(\partial_t - \Delta)u = u\zeta.$$

- The Navier-Stokes equation, also in dimensions 2 and 3 :

$$(\partial_t - \Delta)v = -P(v.\nabla)v + \zeta.$$

- The Sine-Gordon model in dimension 2, for $\beta^2 \in (0, \frac{16\pi}{3})$:

$$(\partial_t - \frac{1}{2}\Delta)u = c\sin(\beta u) + \zeta.$$

In all the examples above, ζ is the space-time white noise but the theory is not limited to the Gaussian case. In this thesis, the Gaussian character of the noise term is only referred to as a practical example for which we know explicitly how to build the renormalised model. The results obtained are deterministic and only the regularity of the noise and of the distributions in the model play a role in the analysis.

1.2.1 Abstract regularity structures

Given a sub-critical singular SPDE, a regularity structure is a way to compute the renormalised equation and to locally describe solutions to this equation. It provides an algebraic framework to describe rough distributions, that replaces or completes the usual polynomial expansion that is suitable to describe smooth functions. These algebraic elements are derived from the equation together with the calculus needed to express a fixed point argument.

A regularity structure is an abstract graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, for a discrete set A with a lower bound α_0, equipped with a structure group G. The basis elements $\mathcal{T}$ of the free vector space T represent the different iterated integrals of the noise and the structure group is used to describe the relations between them.

The elements of $\mathcal{T}$ can be obtained by a formal fixed point argument with an abstract equation. For the Φ^4 equation, we need an abstract heat kernel $\mathcal{I}$ and the abstract noise Ξ to formulate the abstract Φ^4 equation:

$$U = -\mathcal{I}(U^3 + \Xi). \tag{1.2.1}$$

We also want the regularity structure to contain the polynomial structure, so we introduce the abstract monomials $\mathbf{X}_i,\ i = 0, 1...d$, as well as some derivative operators $D_i,\ i = 0, 1...d$. The elements of $\mathcal{T}$ are therefore constructed from:

- A set of generators $\{\mathbf{X}_i,\ i = 1...d,\ \Xi\}$;
- An operator $\mathcal{I}$;
- Derivative operators D_i;
- A product of elements.

These elements are conveniently represented graphically by trees with nodes being the noises and polynomials, where edges represent the operator $\mathcal{I}$ and its derivatives $D_i\mathcal{I}$, joined at the root to denote multiplication. The first step of the iteration gives the trees that we already introduced after (1.1.8).

$$\mathcal{I}(\Xi) = \text{\Large\textbf{!}},\ \mathcal{I}(\Xi)^2 = \text{\Large\textbf{V}},\ \mathcal{I}(\Xi)^3 = \text{\Large\textbf{Y}}.$$

Further trees appear in the next step, for example:

$$\mathcal{I}(\mathcal{I}(\Xi)^3) = \text{\Large\textbf{Y}},\ \mathcal{I}(\Xi)^2(\mathcal{I}(\Xi)^2) = \text{\Large\textbf{X}},\ \mathcal{I}(\Xi)\mathcal{I}(\mathcal{I}(\Xi)^3) = \text{\Large\textbf{Y}}.$$

We associate to each basis element $\tau \in \mathcal{T}$ an order denoted $|\tau|$, which is derived from the regularity of the noise and the gain of regularity from the concrete operators. The space T is then a graded vector space with for every $\tau \in \mathcal{T}$, $\tau \in T_\alpha$ for $\alpha = |\tau|$. In the case of the Φ^4 equation, the homogeneity of Ξ will depend on the dimension in which we want to solve the equation, and the operator $\mathcal{I}$ adds 2. Multiplication of two trees by joining them at the root adds orders. This is different from regularity, since multiplication of a function and a distribution of negative regularity gives in general a distribution of the same negative regularity.

1.2.2 Models

As opposed to our approach discussed previously in Section 1.1.2, those trees do not correspond directly to a distribution. Mapping the abstract space of regularity structure to functions and distributions is the role of the model map denoted $\mathbf{\Pi}$. A requirement for $\mathbf{\Pi}$ is that for $\tau \in \mathcal{T}$, $\mathbf{\Pi}\mathcal{I}(\tau) = \mathcal{L}^{-1}(\mathbf{\Pi}\tau)$ where the operator $\mathcal{L}^{-1}$ is an integration against the heat kernel, or a truncated version of the heat kernel. A model will also be said to be a lift of a noise distribution ζ when it is build based on this noise. We will typically have $\mathbf{\Pi\Xi} = \zeta$. However, there are different ways to define what $\mathbf{\Pi}$ does with products.

A solution $U \in T$ of equation (1.2.1) will then be mapped to a distribution u and different models $\mathbf{\Pi}$ correspond to solving different equations. For example, equation (1.1.6) corresponds to the multiplicative model which does not converge as $\delta \to 0$; equation (1.1.7)

corresponds to the renormalised product for which

$$\mathbf{\Pi}\mathbb{V} = \mathbb{I}_\delta^2 - C_\delta \text{ and } \mathbf{\Pi}\mathbb{V}^2 = \mathbb{I}_\delta^3 - 3C_\delta \mathbb{I}_\delta.$$

We show that these converge as $\delta \to 0$ in Section 1.2.3. In [11], the authors show for a wide class of equations how a change from the multiplicative model is equivalent to solving a modified equation, and a process to determine this modification is also introduced.

The existence of (local) solutions for a given model is a consequence of a series of order bounds. As in the case of polynomials, the homogeneity of an element $\tau \in \mathcal{T}$ describes the behaviour of $\mathbf{\Pi}\tau$ around a base point, for example the origin 0. In order to be able to place this point anywhere, a family of invertible operators $F_x \in G,\ x \in \mathbb{R}^d$ is introduced, such that $\Pi_x(\tau) = \mathbf{\Pi}F_x\tau$ has a behaviour of order $|\tau|$ around the point x. This is quantified by the following bound:

$$|(\Pi_x(\tau) * \Psi_L)(x)| \leqslant C_\tau L^{|\tau|},$$

where Ψ_L is an approximation of unity at scale L. One can then use the maps $\Gamma_{xy} = F_x^{-1} \circ F_y$, which gives the relation $\Pi_y = \Pi_x \circ \Gamma_{xy}$ and $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$.

Here one more feature of the structure group becomes apparent. For an element $\tau \in \mathcal{T}$, we have the local expansion

$$\Pi_y\tau = \Pi_x \circ \Gamma_{xy}\tau = \sum_{\overline{\tau} \in \mathcal{T}} \gamma_\tau^{\overline{\tau}}(x,y)\Pi_x\overline{\tau},$$

for coefficients $\gamma_\tau^{\overline{\tau}}(x,y) \in \mathbb{R}$. The previous order bound translates to a requirement for every element $\Gamma \in G$ to have, for any $a \in T_\alpha$,

$$\Gamma a - a \in \sum_{\beta<\alpha} T_\beta,$$

as well as a quantitative bound [41, Eq (2.15)]

$$|\gamma_\tau^{\overline{\tau}}(x,y)| \lesssim d(x,y)^{|\tau|-|\overline{\tau}|}. \tag{1.2.2}$$

The objective behind the definition of a regularity structure and a model is the definition of the nonlinear operations appearing in the SPDE. They are defined on a class of distributions, called *modelled distributions* whose local behaviour is described by the distributions in the model. The space D_β^γ is defined for any $\gamma \in \mathbb{R}$ as the space of locally bounded functions $U : \mathbb{R}^d \to \bigotimes_{\alpha>\beta} T_\alpha$ with $U(x) = \sum_{|\tau|>\beta} u_\tau(x)\tau$ such that for any

$\tau \in \mathcal{T}$ with $|\tau| < \gamma$,

$$|u_\tau(x) - \sum_{\overline{\tau}} \gamma_\tau^{\overline{\tau}}(x,y) u_{\overline{\tau}}(y)| \lesssim d(x,y)^{\gamma - |\tau|}. \tag{1.2.3}$$

Those represent the functions or distributions that are of regularity β, but can be approximated up to regularity γ by a linear combination of distributions given by the model. Multiplication is then defined for modelled distributions [41, Th 4.7]

$$D^{\gamma_1}_{\alpha_1} \times D^{\gamma_2}_{\alpha_2} \to D^{\gamma}_{\alpha}$$

where $\alpha = \alpha_1 + \alpha_2$ and $\gamma = \min\{\gamma_1 + \alpha_2, \gamma_2 + \alpha_1\}$. If we think about classical Taylor expansion, this is not too different from saying that the multiplication of a C^a function and a C^b function gives a $C^{a \wedge b}$ function, γ being 0 in both cases.

The reconstruction theorem [41, Th 3.10] also guarantees that if $\gamma > 0$ and $U \in D^\gamma_\beta$, there exists a unique distribution $\mathcal{R}U$ in the Hölder space C^β, called the reconstruction of U, such that $\Pi_x U(x) - \mathcal{R}U(x) \in C^\gamma$. The combination of these two theorems gives then a way to define a distribution as the product of two distributions.

In our example from Section 1.1.2, it means the products $v_\delta \varv_\delta$ and $v_\delta^2 \varv_\delta$ will converge in the limit $\delta \to 0$ if we define them by using an expansion of v_δ and v_δ^2 up to homogeneity better than $-|\varv|$ and $-|\varv|$ respectively, as we do directly in Chapter 3 where we bypass the abstract theory and work directly with the few function and distributions $\Pi_x(\tau)$ needed for the Φ^4_3 equation.

1.2.3 Stochastic renormalisation

Renormalisation of singular SPDEs in the framework of regularity structures is a well understood process and [9, 18] presents a systematic approach to the problem. In the case of the Φ^4_3 equation, renormalisation is discussed in great detail in [56], and we will follow this approach in our shorter presentation here. However, we first introduce the idea of renormalisation with one example that does not require any knowledge of stochastic analysis. For locally integrable functions on the real line $\mathbb{R}$, it is possible to define a corresponding distribution using the L^2 product.

The function $x \to \frac{1}{x}$ is not locally integrable, but the principal value gives a way to define a canonical distribution, defining $pv(\frac{1}{x})(\phi)$ as the limit when $\epsilon \to 0$ for $1 > \epsilon > 0$ of

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{\phi(x)}{x} dx = \int_{\mathbb{R}\setminus(-1,1)} \frac{\phi(x)}{x} dx + \int_{[-1,1]\setminus(-\epsilon,\epsilon)} \frac{\phi(x) - \phi(0)}{x} dx,$$

where the equality is a consequence of the antisymmetry of $x \to \frac{1}{x}$. The convergence is assured since for $|x| < 1$, $|\phi(x) - \phi(0)| \leqslant |x| \|\phi'\|_{(0,1)}$. For the function $x \to \frac{1}{|x|}$

however we do not have anti-symmetry, but the quantity

$$\int_{\mathbb{R}\setminus(-1,1)} \frac{\phi(x)}{|x|}dx + \int_{[-1,1]\setminus(-\epsilon,\epsilon)} \frac{\phi(x)-\phi(0)}{|x|}dx \tag{1.2.4}$$

still converges for the same reason. We are therefore tempted to define a "renormalised" distribution for $x \to \frac{1}{|x|}$ with

$$\lim_{\epsilon\to 0}\Big(\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{\phi(x)}{|x|}dx + 2\log(\epsilon)\phi(0)\Big). \tag{1.2.5}$$

There was however no canonical reason to chose the interval $[-1,1]$ in (1.2.4). A different choice of interval containing 0 would also lead to a converging integral, and would result in a different diverging constant in (1.2.5). The rate of divergence $2\log(\epsilon)\phi(0)$ would remain the same however.

The same idea applies when defining the constant C_δ appearing in equation (1.1.7): there is no canonical choice of C_δ, only the rate of divergence of C_δ matters. As an example, we present the case of a centred Gaussian noise on the torus in 2 or 3 dimensions as in (1.1.4), but the scope of the theory of regularity structures in much more general.

In our case we can write:

$$\widehat{\Upsilon}_\delta(t,\omega) = \mathbb{1}_{\{|\omega|<\delta^{-1}\}}\int_{u=-\infty}^{t} \widehat{P}_{t-u}(\omega)dW(u,\omega),$$

where $\widehat{P}_t$ is the Fourier transform of the heat kernel

$$\widehat{P}_t(\omega) = e^{-t(1+4\pi^2|\omega|^2)}.$$

We have then

$$C_\delta := \mathbb{E}[\Upsilon_\delta(0)^2] = \mathbb{E}[\Big(\sum_{|\omega|<\delta^{-1}}\int_{u=-\infty}^{0}\widehat{P}_{-u}(\omega)dW(u,\omega)\Big)^2] \tag{1.2.6}$$

$$= \sum_{|\omega|<\delta^{-1}}\int_{u=-\infty}^{0}|\widehat{P}_{-u}(\omega)|^2du = \sum_{|\omega|<\delta^{-1}}\frac{1}{1+4\pi|\omega|^2}. \tag{1.2.7}$$

This quantity diverges as $\delta\to 0$ with rate proportional to $\log(\delta)$ in dimension 2, and to δ^{-1} in dimension 3.

The following computation using Itô calculus shows that with definition of C_δ, $\mathbf{\Pi}\mathbb{V}_\delta = \Upsilon_\delta^2 - C_\delta$ converges as $\delta\to 0$ to a distribution that we denote $\mathbb{V}$

$$\widehat{\mathfrak{I}}_\delta^2(t,\omega) = \sum_{\omega_1+\omega_2=\omega} \widehat{\mathfrak{I}}_\delta(t,\omega_1)\widehat{\mathfrak{I}}_\delta(t,\omega_2)$$
$$= \sum_{\substack{\omega_1+\omega_2=\omega \\ |\omega_i|<\delta^{-1}}} \Big(\int_{u=-\infty}^t \widehat{P}_{t-u}(\omega_1)dW(u,\omega_1)\Big)\Big(\int_{u=-\infty}^t \widehat{P}_{t-u}(\omega_2)dW(u,\omega_2)\Big)$$
$$= \sum_{\substack{\omega_1+\omega_2=\omega \\ |\omega_i|<\delta^{-1}}} \Big(2\int_{-\infty}^t \Big[\int_{-\infty}^{u_1} \widehat{P}_{t-u_1}(\omega_1)\widehat{P}_{t-u_2}(\omega_2)dW(u_2,\omega_2)\Big]dW(u_1,\omega_1)$$
$$+ \mathbb{1}_{\{\omega_1=-\omega_2\}}\int_{-\infty}^t \widehat{P}_{t-u}(\omega_1)\widehat{P}_{t-u}(\omega_2)du\Big),$$

and we see that summation of the last term gives exactly the one we have subtracted when $\omega = 0$, and vanishes for $\omega \neq 0$. A similar computation shows that $\mathbf{\Pi}\mathfrak{V}_\delta = \mathfrak{I}_\delta^3 - 3C_\delta\mathfrak{I}_\delta$ also converges:

$$\widehat{\mathfrak{I}}_\delta^3(t,\omega) = \sum_{\omega_1+\omega_2+\omega_3=\omega} \widehat{\mathfrak{I}}_\delta(t,\omega_1)\widehat{\mathfrak{I}}_\delta(t,\omega_2)\widehat{\mathfrak{I}}_\delta(t,\omega_3)$$
$$= \sum_{\substack{\omega_1+\omega_2+\omega_3=\omega \\ |\omega_i|<\delta^{-1}}} \Big(\int_{u=-\infty}^t \widehat{P}_{t-u}(\omega_1)dW(u,\omega_1)\Big)\Big(\int_{u=-\infty}^t \widehat{P}_{t-u}(\omega_2)dW(u,\omega_2)\Big)$$
$$\times \Big(\int_{u=-\infty}^t \widehat{P}_{t-u}(\omega_3)dW(u,\omega_3)\Big)$$
$$= \sum_{\substack{\omega_1+\omega_2+\omega_3=\omega \\ |\omega_i|<\delta^{-1}}} \Big(6\int_{-\infty}^t \Big(\int_{-\infty}^{u_1}\Big(\int_{-\infty}^{u_2} \widehat{P}_{t-u_1}(\omega_1)\widehat{P}_{t-u_2}(\omega_2)\widehat{P}_{t-u_3}(\omega_3)dW(u_3,\omega_3)\Big)$$
$$\times dW(u_2,\omega_2)\Big)dW(u_1,\omega_1)$$
$$+ 3\mathbb{1}_{\{\omega_1=-\omega_2\}}\widehat{\mathfrak{I}}_\delta(t,\omega_3)\int_{-\infty}^t \widehat{P}_{t-u}(\omega_1)\widehat{P}_{t-u}(\omega_2)du\Big).$$

For the Φ^4 equation in dimension 3 as in Chapter 3, one more renormalisation constant is needed. It is defined by

$$C'_\delta := \mathbb{E}[\mathfrak{V}_\delta(0)\mathcal{L}^{-1}\mathfrak{V}_\delta(0)]$$

where $\mathfrak{V}_\delta$ is the distribution constructed previously. We have

$$\mathbb{E}[\mathfrak{I}_\delta^2(0)\mathcal{L}^{-1}(\mathfrak{I}_\delta^2(0))] =$$
$$\mathbb{E}[\sum_{\substack{|w_i|<\delta^{-1} \\ \omega_3+\omega_4+\omega_5=0}} \int_{u_3,u_4,u_5=-\infty}^0 \widehat{P}_{-u_3}(\omega_3)\widehat{P}_{-u_4}(\omega_4)\widehat{P}_{-u_5}(\omega_5)$$

$$\sum_{\omega_1+\omega_2=\omega_5} \int_{u_1,u_2=-\infty}^{u_5} \widehat{P}_{u_5-u_1}(\omega_1)\widehat{P}_{u_5-u_2}(\omega_2) dW(u_1,\omega_1)dW(u_2,\omega_2)$$
$$dW(u_3,\omega_3)dW(u_4,\omega_4)du_5].$$

We use a formula discovered by Isserlis and Wick [47] to compute this expectation involving a product of several Gaussian variables. This formula states that for an even number of Gaussian random variables, the expectation can be computed by adding all the ways of matching pairs of random variables and taking their covariances. Here, matching variables $W(u_1,\omega_1)$ with $W(u_2,\omega_2)$ and $W(u_3,\omega_3)$ with $W(u_4,\omega_4)$ corresponds to the quantities we have already subtracted in the construction of $\mathbb{V}$. There are two symmetric cross-terms remaining, and the expectation vanishes except for when $\omega_1 = -\omega_3$, and $\omega_2 = -\omega_4$, which gives the following value for C'_δ diverging with rate $\log(\delta)$ as $\delta \to 0$:

$$\begin{aligned}
C'_\delta =& 2 \sum_{\substack{|w_i|<\delta^{-1} \\ \omega_1+\omega_2=\omega_5}} \int_{u_5=-\infty}^{0} \widehat{P}_{-u_5}(\omega_5) \\
& \int_{u_1,u_2=-\infty}^{u_5} \widehat{P}_{u_5-u_1}(\omega_1)\widehat{P}_{-u_1}(-\omega_1)\widehat{P}_{u_5-u_2}(\omega_2)\widehat{P}_{-u_2}(-\omega_2)du_1du_2du_5 \\
=& 2 \sum_{\substack{|w_i|<\delta^{-1} \\ \omega_1+\omega_2=\omega_5}} \int_{-\infty}^{0} P_{-u_5}(\omega_5)du_5 \frac{e^{-u_5(1+4\pi|\omega_1|^2)}}{2(1+4\pi|\omega_1|^2)} \frac{e^{-u_5(1+4\pi|\omega_2|^2)}}{2(1+4\pi|\omega_2|^2)} \\
=& \frac{1}{2} \sum_{\substack{|w_i|<\delta^{-1} \\ \omega_1+\omega_2=\omega_5}} \frac{1}{1+4\pi|\omega_1|^2} \frac{1}{1+4\pi|\omega_2|^2} \frac{1}{3+4\pi(|\omega_1|^2+|\omega_2|^2+|\omega_5|^2)}.
\end{aligned}$$

1.3 From small scales to large scales

The purpose of the Chapter 2 was in part to understand how to link large scale bounds, using the effect of the strong damping $-u^3$ and the small scale analysis done in this simple case by Schauder estimates, and in more complicated cases with tools inspired from the theory of regularity structures.

1.3.1 The maximum principle

It is well-known that if f satisfies the so-called Osgood condition, that is if f satisfies $\int_1^\infty \frac{1}{f(u)}du < \infty$, then solutions of the ODE $\dot{x} = -f(x)$ "come down from infinity in finite time" (see [59]). This means that if x solves the equation over $[0,t]$, then automatically $x(t)$ satisfies a bound which depends on t, but holds uniformly over all possible choices of initial datum $x(0) > 0$. Similar statements can be derived for reaction diffusion equations based on a comparison principle (see e.g. [66, Chapter 14]) and also

stochastic reaction diffusion equations (see e.g. [15, Theorem 6.2.3] and [16]). These bounds are powerful tools to study the long-time behaviour of solutions, both in the deterministic and in the stochastic setting - see e.g. [68] for a construction of invariant measures for stochastic PDEs based on such bounds.

The maximum principle that allowed for large scale results for the equation (1.0.1) is introduced in the first chapter as Theorem 2.4.4. In this first chapter, we even have a more elaborate version which allows us to treat more general nonlinearities, but the idea behind the proof is the same.

Analysis of a few examples suggests that this bound is of order $\Theta^{-1}\big(\frac{1}{t}\big)$ where $\Theta(x) = \frac{f(x)}{x}$. One may think for example of the case $\dot{x} = -x^3$, which has for solution $x(t) = (2t + x_0^{-2})^{-\frac{1}{2}} < (2t)^{-\frac{1}{2}}$.

In order to prove a similar bound in space-time for reaction diffusion equations of the type

$$(\partial_t - \Delta)u = -f(u) + g, \tag{1.3.1}$$

where f is as before and g is bounded, we multiply the equation with a smooth function η compactly supported on a domain D and we derive from the equation (1.3.1) a set of inequalities that the product $u\eta$ satisfies with $u\eta < 1$. To finish the proof, we needed to find η that was a smooth version of $\Theta^{-1}\big(\frac{1}{d(\cdot,\partial D)}\big)$, and that satisfied the inequalities. The weakest condition we obtained in Theorem 2.4.4 was not quite the Osgood condition, but slightly stronger. We believe our condition may still be sharp in this PDE setting.

This maximum principle does not apply directly to the Φ^4 equation, because it requires a bounded right-hand side. In the first publication we find a way around this in the non-singular case where $\zeta \in C^{\alpha-2}$ for some positive α. The technique adopted consists of convolving the equation with a smooth, compactly supported regularising kernel Ψ, rescaled at some level L. This gives a bound on $u * \Psi_L$ and the difference between $u(z)$ and $u * \Psi_L(z)$ is controlled by the α Hölder regularity of u on $B(z, L)$. The small enlargement of the domain due to this convolution requires the choice of a specific L and then to iterate the argument.

The choice of L can be deduced from computations but there is also a heuristic argument that suggests it should be proportional to $\Theta^{-1}(\|u\|_{\{x,d(x,\partial D)>t\}})$. It also explains why in Chapter 2, we were not able to go beyond the polynomial case despite having a more powerful maximum principle. The bound that comes out of the maximum principle is $\Theta^{-1}\big(\frac{1}{d(\cdot,\partial D)}\big)$, and it is reasonable to think that only quantities related to $\|u\|_{\{x,d(x,\partial D)>t\}}$ should be appearing in the bound on $u - u * \Psi_L$. The quantity $\Theta^{-1}(\frac{x}{2})$ is comparable to $\Theta^{-1}(x)$ when f is a polynomial, which is what allows one to close the argument in that case.

In the other chapters we work in the case when $\zeta \in C^{\alpha-2}$ for $\alpha < 0$, and therefore the expected regularity for u is negative. This means that $u \in C^\alpha$ is not the quantity we want to look at. Instead, we look at the remainder equation in the manner of (1.1.8) and treat the other terms as many other bounded terms, but the part of the analysis where we apply the maximum principle is essentially the same.

1.3.2 Main results

In this whole thesis, we never address the issue of existence of solution to the equation (1.0.1). Even in the most irregular case, we consider smooth approximations of this equation, for example taking a sequence of noise terms (or models when regularity structures are needed) $(\xi_n)_{n\in\mathbb{N}}$ where each element ξ_n is smooth, but the sequence converges only in the low regularity space in which we measure the distribution. We do not prove convergence of the solutions, but our results hold uniformly in the parameter n, which remains implicit in all the thesis. This implies in particular that we also do not concern ourselves with knowing in which sense the equation holds. We give precisions on this topic at the beginning of section 3.2.2. We assume that the equation holds on a neighbourhood of a domain D, and we only require information on the noise or model on a 1-enlargement of this domain.

The bounds we obtain in the end in Theorems 2.3.1,3.2.1 and 4.9.1 provide then a space-time "coming down from infinity" on any domain for what we may call the regular part of the solution (u in dimension 1, $v = u - \varpi$ in dimension 2...), or a bound in terms of the noise and the functions and distributions in the model:

$$\|v\|_R \leqslant C \max\{\frac{1}{R}, [\mathbf{\Pi}\tau]^{\frac{1}{\delta n(\tau)}}, \tau \in \mathcal{T}\}, \tag{1.3.2}$$

where v represents u in dimension 1, $u - \varpi$ in dimension 2 and so on. $\| \bullet \|_R$ is the L^∞ norm on a domain $D_R = D \setminus \{\mathbf{X} \in D, d(x, \partial D) \leqslant R\}$ and $[\mathbf{\Pi}\tau]$ is the $C^{|\tau|}$ norm of $\Pi_\bullet(\tau)$. The set $\mathcal{T}$ and the exponent δ depend on the regularity of the noise. The exponent $n(\tau)$ represents in the case of Gaussian noise the Wiener chaos of highest degree in which $\Pi_\bullet(\tau)$ lies. This bound gives a bound on a solution on the full space at any space-time point that depends only on the realisation of the noise and the other terms in the model on a compact space around this point.

This is stronger than the bound in [35], where a bound on the solution on weighted Besov spaces is obtained. Furthermore, in [35] and in all previous works on global solutions for the dynamic Φ^4 equation [36, 54, 55, 70], the local solution theory was done using paracontrolled calculus, which is another approach to singular SPDEs [38]. The theory of regularity structures has so far been expressed in a more general setting and for example the analysis done in Chapter 4 is out of reach of the current status of the

theory of paracontrolled calculus.

Furthermore, one of the contributions of Chapter 4 to the theory of regularity structures is the construction of a model that only depends locally on the noise. Instead of using a kernel approach to describe the integration $\mathcal{L}^{-1}(\tau)$ of a distribution τ, we only impose that it is solution to a PDE with cutoff on the right-hand side. We also use the same ad-hoc localised Schauder theory for singular equations developed in Chapter 3.

One corollary of the main result is the existence of moment bounds for the remainder part of the solution. The homogeneity of the terms in the bound (1.3.2) is such that in the case of Gaussian white noise, the exponential stochastic bounds that can be derived form each of these terms coincide. We get in that case the existence of a constant λ such that:

$$\mathbb{E}[\exp(\lambda \|v\|^{2\delta})] < \infty,$$

where the parameter δ depends on the regularity of the noise, or equivalently on the dimension. This gives us reason to think our result is optimal, and once this was understood, it also allowed for a simplification of the proof by assuming that the bounds in terms of the model in equation (1.3.2) *did not hold* (equation (3.4.1) and Assumption 4.9.6) and using this assumption to prove the localisation part of the bound.

Chapter 2

Local bounds for stochastic reaction diffusion equations

2.1 Introduction

In this chapter we are interested in reaction diffusion equations of the type

$$(\partial_t - \Delta)u = -f(u) + \zeta, \tag{2.1.1}$$

over $\mathbb{R}_t \times \mathbb{R}^d_x$ where ζ is an irregular distribution of regularity $\alpha - 2$ for some $\alpha > 0$. The example we have in mind is the case where ζ is a random noise term, such as space-time white noise for $d = 1$, or a noise which is "white in time and coloured in space" for $d \geqslant 2$. However, we mention once again that our main result is purely deterministic and the only information about ζ that enters is its regularity measured in a suitable space of distributions. The nonlinearity f is assumed to be continuous, with super-linear growth at infinity in u.

Our main result of this chapter, Theorem 2.3.1, is a space-time comming down from infinity bound for solutions of (2.1.1) with $f(u) = u|u|^{m-1} + g$ where g is bounded. We consider a continuous function $u\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and we assume that (2.1.1) holds for (t, x) in a cylinder, say

$$P_0 := (0, 1) \times (-1, 1)^d.$$

Then for $R < \frac{1}{2}$ the L^∞ norm $\|u\|_{P_R}$ of u on the unit cylinder minus a parabolic boundary layer of size R

$$P_R := (R^2, 1) \times (-(1 - R), 1 - R)^d, \tag{2.1.2}$$

satisfies a bound which only depends on R and a distributional norm of ζ restricted to the original cylinder P_0:

$$\|u\|_{P_R} \leqslant C(\alpha, d, m) \max\Big\{R^{-\frac{2}{m-1}}, [\zeta]_{\alpha-2,P_0}^{\frac{2}{2+(m-1)\alpha}}, \|g\|^{\frac{1}{m}}\Big\}, \tag{2.1.3}$$

where $[\zeta]_{\alpha-2,P_0}$ is the space-time Hölder norm of order $\alpha-2$ on P_0 (see (2.3.7) below for a precise definition), and $\|g\|$ refers to the supremum norm of g.

One possible application of the bound (2.1.3) is the *construction* of solutions to (2.1.1) on the full space. The standard approach to solve stochastic reaction diffusion equations [24, 39, 71] consists of writing the equation in its mild form and solving the corresponding fixed point problem using Picard iterations. However, this approach requires a pathwise uniform-in-x control on ζ, which typically only holds on bounded domains or if ζ decays at ∞; the interesting case of spatially stationary noise cannot be treated directly in this way. This problem was overcome in [48] where solutions were first constructed on a sequence of growing tori and then a compactness argument in a space with weights was used to pass to the limit. The strong localisation obtained in (2.1.3) should allow for a significant simplification of this construction.

The estimate (2.1.3) also has an interesting consequence for the stochastic integrability of u. In fact, we are mostly interested in the case where ζ is a random distribution with Gaussian tails such that $\mathbb{E}\big[e^{\varepsilon[\zeta]_{\alpha-2,P_0}^2}\big]$ is finite for $\varepsilon > 0$ small enough. The estimate (2.1.3) then immediately implies that for any $R > 0$ and for $\varepsilon > 0$ small enough we get $\mathbb{E}\big[e^{\varepsilon\|u\|_{P_R}^{2+(m-1)\alpha}}\big] < \infty$. So $\|u\|_{P_R}$ has lighter tails than Gaussian. We observe that better pathwise *regularity* for ζ leads to better *integrability* with respect to the probability distribution for u. In the special case of one-dimensional reaction-diffusion equations where ζ is a space-time white noise, equation (2.1.1) equipped with suitable boundary conditions defines a reversible Markov process, and an explicit expression of the equilibrium measure is available. In Section 2.7 we argue that in this case the integrability we derive from estimate (2.1.3) coincides with the integrability derived from the explicit invariant measure.

Finally, our method shows in this chapter a new perspective on singular SPDE with the convolution argument. Our starting point is Hairer's notion [41] of *subcriticality* which in the context of (2.1.1) states, roughly speaking, that the *small scale* behaviour of solutions should be determined by the interplay of the heat operator and the rough driving noise ζ, while on *large scales* the nonlinearity becomes dominant. We implement this philosophy by regularising (2.1.1) on a lengthscale L by convolving the equation

with a suitable regularising kernel, arriving at

$$(\partial_t - \Delta)u_L = -u_L|u_L|^{m-1} + g(u)_L + \zeta_L + [u_L|u_L|^{m-1} - (u|u|^{m-1})_L], \quad (2.1.4)$$

where the subscript L denotes a regularised quantity. The extra term $[u_L|u_L|^{m-1} - (u|u|^{m-1})_L]$ on the right hand side appears because regularisation and application of the polynomial do not commute. We then use a low regularity version of classical Schauder theory, Lemma 2.9.1, to control the error term $[u_L|u_L|^{m-1} - (u|u|^{m-1})_L]$. Using this bound, the remaining terms can be treated as in the smooth case (see Theorem 2.4.4).

In the more regular case presented here it would be natural to aim to also include more general nonlinearities, such as functions with faster than polynomial growth (e.g. the $\exp(\phi)$ model $f(u) = \sinh(u)$, see [3]) or functions of slower than polynomial growth such as $f(u) \sim u\log(u)^\delta$ for $\delta > 2$. In this case the commutator term arising in (2.1.4) turns into

$$f(u_L) - f(u)_L.$$

Unfortunately, our method crucially depends on the fact that $xf'(x) \lesssim f(x)$ which holds for polynomial f, but not for functions with exponential growth. Also, another part of our argument excludes functions that grow to slowly (in the proof of Theorem 2.3.1 we need to sum $\Theta(u) := \frac{f(u)}{u}$ for $u = 2^{-k}$, $k \in \mathbb{N}$), thus essentially restricting us to polynomial f. However, in the case of a more general nonlinearity f, we implement a more standard argument based on subtracting the solution w to the linear equation

$$(\partial_t - \Delta)w = \zeta,$$

and we do not pass through the regularised equation (2.1.4). We then get the property of "coming down from infinity" for the remainder $u - w$ in Corollary 2.4.6. For example, when $f(u) = \sinh(u)$, the strong damping implies that u "comes down from infinity" much more quickly than in the polynomial case – in this case the function $R^{-\frac{2}{m-1}}$ in (2.1.3) turns into $\Theta^{-1}(R^{-2})$, where $\Theta(R) = \frac{\sinh(R)}{R}$. For very weak damping $f(u) \sim u\log(u)^{\alpha+2}$, we obtain a slow coming down from infinity, of order $\exp(R^{-\frac{2}{\alpha}})$. In fact, this method is even easier than the method we use for the polynomial case, but it has two significant disadvantages: On the one hand, it is impossible to measure the fine interplay between regularity of ζ and integrability of u in this way, because the remainder $u - w$ can never have better integrability than the Gaussian process w. More importantly, the more sophisticated method we use in the proof of our main theorem is crucial when dealing with more singular equations in the following chapters.

The rest of the chapter is structured as follows. In Section 2.2 we discuss the elementary

case of the stochastic ODE

$$dx(t) = -|x(t)|^{m-1}x(t)dt + dw(t),$$

in which our strategy and also the interplay between the regularity of the noise and integrability of the solution becomes apparent in a technically simple context. In Section 2.3 we introduce the framework and state the main result. The proof is split into Sections 2.4–2.5: In Section 2.4 we present a proof of the "space-time coming down from infinity" in the case where ζ is replaced by a smooth function. This result is an essential ingredient also in the following chapters. The argument relies on a maximum principle. As a corollary, as discussed above, we derive the bounds on the remainder $u - w$ in the case of general, not necessarily polynomial f. In Section 2.5 the result of Section 2.4 is applied to the regularised equation (2.1.4) and combined with Schauder estimates to bound the commutator concluding the proof of our main result. In Section 2.6 we discuss the case of a random distribution ζ given by the time-derivative of the stochastic integral $\int_0^t \sigma dW$ for an adapted bounded process $\sigma = \sigma(s, x)$ and a distribution valued Wiener process W with suitable (spatial) covariance operator. We show Gaussian estimates for $[\zeta]_{\alpha-2}$ and thus better than Gaussian bounds for u. Finally, in the special case of space-time white noise in one spatial dimension we show that the integrability obtained from our method coincides with the integrability of the process in equilibrium obtained from the explicit invariant measure.

2.2 The ODE case

Before dealing with equation (2.1.1) we briefly discuss the case of a (stochastic) ordinary differential equation

$$dx(t) = -|x(t)|^{m-1}x(t)dt + dw(t) \tag{2.2.1}$$

for a standard Brownian motion $w(t)$ and for $m > 1$. It is well known that (2.2.1) defines a reversible Markov process with respect to the measure

$$\mu(dx) \propto \exp\Big(-\frac{2}{m+1}|x|^{m+1}\Big)dx. \tag{2.2.2}$$

We seek to derive optimal bounds on solutions of $x(t)$ directly from the equation (2.2.1).

As a starting point, consider the case of an ordinary differential equation driven by a regular noise term η

$$\dot{x}(t) = -x(t)|x(t)|^{m-1} + \eta(t). \tag{2.2.3}$$

A simple ODE comparison Lemma, see [68, Lemma 3.8], shows that for $t \in (0,1]$

$$|x(t)| \leqslant C(m) \max\left\{ t^{-\frac{1}{m-1}}, \Big(\sup_{t\in[0,1]} |\eta(t)| \Big)^{\frac{1}{m}} \right\}, \tag{2.2.4}$$

uniformly over all choices of initial datum $x(0)$. If η is a Gaussian process, such that the random variable $\sup_{t\in[0,1]} |\eta(t)|$ has finite Gaussian moments, this bound implies that for $\varepsilon > 0$ small enough

$$\mathbb{E}\left[\exp(\varepsilon |x(1)|^{2m})\right] < \infty. \tag{2.2.5}$$

In particular, in this regular case we get much better integrability than under the measure (2.2.2). The following deterministic lemma shows that the difference in integrability is closely related to the regularity of the driving signal.

As usual, we define the α-Hölder semi-norm as

$$[w]_\alpha = \sup_{0\leqslant s<t\leqslant 1} \frac{|w(t) - w(s)|}{|t-s|^\alpha}.$$

In an attempt to make this proof as similar as possible to the one of our main result, Theorem 2.3.1, we relabel the regularity parameter as $\frac{\alpha}{2}$ for $\alpha \in (0,2)$.

Lemma 2.2.1. *Let $w\colon [0,1] \to \mathbb{R}$ be $\frac{\alpha}{2}$-Hölder continuous for some $\alpha \in (0,2)$ with $w(0) = 0$. For some $m > 1$ let $x\colon [0,1] \to \mathbb{R}$ be a continuous solution to*

$$x(t) = x(0) - \int_0^t |x(s)|^{m-1} x(s) ds + w(t). \tag{2.2.6}$$

Then for $t \in (0,1]$

$$|x(t)| \lesssim \max\left\{ t^{-\frac{1}{m-1}}, [w]_{\frac{\alpha}{2}}^{\frac{2}{2+(m-1)\alpha}} \right\}. \tag{2.2.7}$$

Here and in the proof we use the symbol $\lesssim$ for $\leqslant C(\alpha, m)$.

If w is a random function for which $[w]_\alpha$ has Gaussian tails this estimate yields

$$\mathbb{E}\left[\exp\left(\varepsilon |x(1)|^{2+(m-1)\alpha} \right) \right] < \infty,$$

for ε small enough. In the Brownian case where $\alpha = 1-$ (α can be any value strictly less than 1) the exponent $2 + (m-1)\alpha$ becomes $1 + m-$ in line with (2.2.2) and as α approaches 2, the exponent becomes $2m$ in line with (2.2.5).

Proof of Lemma 2.2.1. The proof follows the same steps, as the proof of the PDE result, Theorem 2.3.1, even though most are considerably simpler.

STEP 1: Local Schauder estimate.

In this context a "Schauder estimate" is trivially derived, simply by writing for $0 < t_1 < t_2 < 1$

$$\begin{aligned}|x(t_2) - x(t_1)| =& \Big| \int_{t_1}^{t_2} x(s)|x(s)|^{m-1} ds + w(t_2) - w(t_1) \Big| \\ \leqslant& |t_2 - t_1| \|x\|^m_{[t_1,t_2]} + |w(t_2) - w(t_1)|,\end{aligned}$$

which can be restated as

$$[x]_{\frac{\alpha}{2},(s-L,s)} \leqslant L^{1-\frac{\alpha}{2}} \|x\|^m_{(s-L,s)} + [w]_{\frac{\alpha}{2}}. \tag{2.2.8}$$

STEP 2: Application of a comparison lemma.

We regularise equation (2.2.6) by convolution. To this end we introduce a smooth non-negative kernel $\Psi \colon \mathbb{R} \to \mathbb{R}$ which is compactly supported in $[0,1]$ with $\int \Psi = 1$ and set $\Psi_L(t) = \frac{1}{L}\Psi(\frac{t}{L})$. For any function $f \colon (0,1) \to \mathbb{R}$ and for $t \in (L,1)$ we define the regularisation $f_L(t) = f * \Psi_L(t) = \int_{t-L}^t \Psi_L(t-s) f(s) ds$.

Convolving the integral equation (2.2.6) with Ψ_L and taking a time derivative leads to

$$\dot{x}_L(t) = -x_L(t)|x_L(t)|^{m-1} + \dot{w}_L(t) + [\cdot^m, (\cdot)_L]x(t) \qquad \text{for } t \in (L,1), \tag{2.2.9}$$

where we write $[\cdot^m, (\cdot)_L]x = \big[x_L|x_L|^{m-1} - (x|x|^{m-1})_L\big]$ for the commutator term on the right hand side.

Now we can apply the ODE comparison result (2.2.4) to get, for all $t \in (L,1]$

$$|x_L(t)| \lesssim \max\Big\{(t-L)^{-\frac{1}{m-1}}, \Big(\sup_{[L,1]} |\dot{w}_L|\Big)^{\frac{1}{m}}, \Big(\sup_{[L,1]} \big|[\cdot^m, (\cdot)_L]x\big|\Big)^{\frac{1}{m}}\Big\}. \tag{2.2.10}$$

STEP 3: Bound on the commutator.

To replace x_L by x and to bound the commutator term on the right hand side, we use the information on the regularity of x provided by Step 1. Indeed, using the fact that Ψ has integral 1, we first see for $t \in (L,1]$,

$$\begin{aligned}|(x_L - x)(t)| =& \Big| \int_{t-L}^t \Psi_L(t-s)(x(s) - x(t)) ds \Big| \\ \leqslant& [x]_{\frac{\alpha}{2},(t-L,t)} \int_{t-L}^t \Psi_L(t-s)|s-t|^{\frac{\alpha}{2}} ds \leqslant L^{\frac{\alpha}{2}} [x]_{\frac{\alpha}{2},(t-L,t)},\end{aligned} \tag{2.2.11}$$

where $[x]_{\frac{\alpha}{2},I} = \sup_{s \neq t \in I} \frac{|x(t)-x(s)|}{|t-s|^{\frac{\alpha}{2}}}$ denotes the $\frac{\alpha}{2}$-Hölder semi-norm of x restricted to

the interval I. Similarly we establish a bound on the commutator: for $s \geqslant L$,

$$|[\cdot^m, (\cdot)_L]x(s)| \lesssim \|x\|^{m-1}_{(s-L,s)} L^{\frac{\alpha}{2}} [x]_{\frac{\alpha}{2},(s-L,s)}, \tag{2.2.12}$$

where $\|x\|_I$ is the supremum norm of x restricted to the interval I. To see (2.2.12) we first write

$$|[\cdot^m, (\cdot)_L]x(s)| = \Big| \int_{s-L}^{s} \Psi_L(s-r)\big(x_L(s)|x_L(s)|^{m-1} - x(r)|x(r)|^{m-1}\big)dr \Big|.$$

Then, using the mean value theorem and $|x_L(s)| \leqslant \|x\|_{(s-L,s)}$, we have

$$|x_L(s)|x_L(s)|^{m-1} - x(r)|x(r)|^{m-1}| \leqslant m\|x\|^{m-1}_{(s-L,s)} |x_L(s) - x(r)|.$$

Finally, using the triangle inequality in the form $|x_L(s) - x(r)| \leqslant |x_L(s) - x(s)| + |x(s) - x(r)| \leqslant 2L^{\frac{\alpha}{2}} [x]_{\frac{\alpha}{2},(s-L,s)}$, we arrive at (2.2.12).

STEP 4: Post processing.

Concerning the noise term on the right hand side of (2.2.10) we write

$$\sup_{t\in[L,1]} |\dot{w}_L(t)| = \sup_{t\in[L,1]} \Big| \int_{t-L}^{t} \dot{\Psi}_L(t-s)\big(w(s) - w(t)\big)ds \Big| \lesssim L^{\frac{\alpha}{2}-1} [w]_{\frac{\alpha}{2}}. \tag{2.2.13}$$

Combining (2.2.10), (2.2.12), (2.2.8) and (2.2.13) we arrive at

$$\begin{aligned} |x_L(t)| \lesssim \max\Big\{ &(t-L)^{-\frac{1}{m-1}}, \Big(L^{\frac{\alpha}{2}-1}[w]_{\frac{\alpha}{2}}\Big)^{\frac{1}{m}}, \Big(L\|x\|^{2m-1}_{(t-L,t)}\Big)^{\frac{1}{m}}, \\ &\Big(L^{\frac{\alpha}{2}}\|x\|^{m-1}_{(t-L,t)}[w]_{\frac{\alpha}{2}}\Big]\Big)^{\frac{1}{m}}\Big\}, \qquad t > L. \end{aligned}$$

Combining this estimate with (2.2.11) and (2.2.8) this estimate turns into

$$\begin{aligned} |x(t)| \lesssim \max\Big\{ &(t-L)^{-\frac{1}{m-1}}, \Big(L^{\frac{\alpha}{2}-1}[w]_{\frac{\alpha}{2}}\Big)^{\frac{1}{m}}, \Big(L\|x\|^{2m-1}_{(t-L,t)}\Big)^{\frac{1}{m}}, \\ &\Big(L^{\frac{\alpha}{2}}\|x\|^{m-1}_{(t-L,t)}[w]_{\frac{\alpha}{2}}\Big]\Big)^{\frac{1}{m}}, L\|x\|^m, L^{\frac{\alpha}{2}}[w]_{\frac{\alpha}{2}}\Big\}, \qquad t > L. \end{aligned}$$

STEP 5: Choosing L.

We choose $L = \frac{\mu}{\|x\|^{m-1}_{(0,1)}}$ for $\mu = \mu(\alpha, m) > 0$ small enough and consider t satisfying $(t-L)^{-\frac{1}{m-1}} \leqslant \frac{1}{2}\|x\|_{(0,1)}$. Then applying Young's inequality $xy \leqslant \delta x^p + C(\delta)y^{p'}$ for $\delta > 0$ and $p, p' \in (0,1)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ multiple times yields

$$\|x\|_{((2^{m-1}+\mu)\|x\|^{1-m}_{(0,1)},1)} \leqslant \max\Big\{ \frac{1}{2}\|x\|_{(0,1)}, C[w]^{\frac{2}{2+(m-1)\alpha}}_{\frac{\alpha}{2}} \Big\}, \tag{2.2.14}$$

for some constant $C = C(\alpha, m)$. Note that we can assume that $(2^{m-1}+\mu)\|x\|_{(0,1)}^{1-m} < 1$, because else we trivially have a bound on $\|x\|_{(0,1)}$.

STEP 6: Iterating the result.

We now define a finite set $0 = t_0 < \ldots < t_N = 1$ by setting $t_{n+1} - t_n = (2^{m-1} + \mu)\|x\|_{(t_n,1)}^{1-m}$ as long as the time t_{n+1} defined this way stays strictly less than 1. We terminate the sequence, once this algorithm would produce a $t_{n+1} \geqslant 1$ in which case we set $t_{n+1} = t_N = 1$. Note that $(2^{m-1} + \mu)\|x\|_{(t_n,1)}^{1-m}$ is increasing in n so the sequence necessarily terminates after finitely many steps.

Applying (2.2.14) to the equation restarted at the times t_n we obtain for $n \leqslant N-1$

$$\|x\|_{(t_n,1)} \leqslant \max\Big\{\frac{1}{2}\|x\|_{(t_{n-1},1)}, C[w]_{\frac{\alpha}{2}}^{\frac{2}{2+(m-1)\alpha}}\Big\}. \tag{2.2.15}$$

We now show that for $n < N$

$$\|x\|_{(t_n,1)} \lesssim \max\Big\{t_{n+1}^{-\frac{1}{m-1}}, [w]_{\frac{\alpha}{2}}^{\frac{2}{2+(m-1)\alpha}}\Big\}. \tag{2.2.16}$$

When the maximum in (2.2.15) is realised by $C[w]_{\frac{\alpha}{2}}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}$, then this follows immediately.

Else, we have for $k \leqslant n$, $\|x\|_{(t_n,1)} \leqslant \|x\|_{(t_k,1)} 2^{k-n}$ and hence

$$\begin{aligned} t_{n+1} =& \sum_{k=0}^{n} t_{k+1} - t_k = (2^{m-1}+\mu)\sum_{k=0}^{n} \|x\|_{(t_k,1)}^{1-m} \\ \leqslant& (2^{m-1}+\mu)\|x\|_{(t_n,1)}^{1-m} \sum_{k=0}^{n} 2^{(n-1-k)(1-m)} \lesssim \|x\|_{(t_n,1)}^{1-m}, \end{aligned} \tag{2.2.17}$$

establishing (2.2.16). For the end point t_N we have either $t_{N-1} \geqslant \frac{1}{2}$ or $t_N - t_{N-1} \geqslant \frac{1}{2}$. In the first case we invoke (2.2.17) for $n = N-1$ and in the second case the definition of $t_{n+1} - t_n$, in both cases yielding the existence of a constant C such that

$$\|x\|_{(t_{N-1},1)}^{1-m} \geqslant C \Rightarrow \|x\|_{(t_{N-1},1)} \leqslant C^{\frac{1}{1-m}},$$

so (2.2.16) also holds for $n = N$. Finally, for intermediate points $t \in [t_{n-1}, t_n]$ we write

$$\begin{aligned} |x(t)| \leqslant \|x\|_{(t_{n-1},1)} &\overset{(2.2.16)}{\lesssim} \max\Big\{t_n^{-\frac{1}{m-1}}, [w]_{\frac{\alpha}{2}}^{\frac{2}{2+(m-1)\alpha}}\Big\} \\ &\leqslant \max\Big\{t^{-\frac{1}{m-1}}, [w]_{\frac{\alpha}{2}}^{\frac{2}{2+(m-1)\alpha}}\Big\}, \end{aligned}$$

so (2.2.7) follows. □

2.3 Setting and main result

After this short interlude, we now go back to the parabolic equation (2.1.1). As usual when dealing with parabolic equations, regularity will be measured with respect to the metric

$$d((t,x),(\bar{t},\overline{x})) = \max\Big\{|x-\overline{x}|, \sqrt{|t-\bar{t}|}\Big\}, \tag{2.3.1}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. We introduce the parabolic ball of centre $z=(x,t)$ and radius R in this metric d, looking only into the past:

$$B(z,R) = \{\overline{z} = (\bar{t},\overline{x}) \in \mathbb{R}\times\mathbb{R}^d,\, d(z,\overline{z}) < R, \bar{t} < t\}. \tag{2.3.2}$$

Recall that P_R is the cylinder at distance R from P_0, as introduced in (2.1.2). Note that for $R' < R$ we have $P_{R'} + B(0,R'-R) \subset P_R$.

For $\alpha \in (0,1)$, we define the Hölder semi-norm $[.]_\alpha$

$$[u]_\alpha := \sup_{z\neq\overline{z}\in\mathbb{R}\times\mathbb{R}^d} \frac{|u(z)-u(\overline{z})|}{d(z,\overline{z})^\alpha}. \tag{2.3.3}$$

We will often deal with local quantities: If $B \subset \mathbb{R}\times\mathbb{R}^d$ is a bounded set, then we define the local α-Hölder semi-norm $[.]_{\alpha,B}$ as in (2.3.3) with the supremum restricted to $z,\overline{z} \in B$. Similarly, $\|.\|$ denotes the supremum norm on the whole space $\mathbb{R}\times\mathbb{R}^d$ and $\|.\|_B$ the supremum norm over B.

To measure distributions in negative Hölder spaces, we introduce a family of mollification operators $\{(.)_L\}$ which are consistent with the scaling given by the heat operator $(x,t) = (l\overline{x}, l^2\overline{t})$. For this we fix a non-negative smooth function Ψ with support in $-B(0,1)$ with $\Psi(z) \in [0,1]$ for all z and with integral 1 and for $L \in (0,1]$ set $\Psi_L(x,t) = \frac{1}{L^{d+2}}\Psi\left(\frac{x}{L}, \frac{t}{L^2}\right)$. We define the operator $(\cdot)_L$ by convolution with Ψ_L, noting that for any L, $(\cdot)_L$ is a contraction on with respect to $\|\cdot\|$. We wish to keep track of the support of the relevant functions. Since Ψ_L is compactly support in $-B(0,L)$,

$$\|h_L\|_K \leqslant \|h\|_{K+B(0,L)} \tag{2.3.4}$$

for any bounded set K. Furthermore, we mention the estimate

$$\int |\Psi_L(x-y)| d(x,y)^\alpha dy \leqslant L^\alpha, \tag{2.3.5}$$

which, as in (2.2.11) above, immediately implies that for any $h \in C^\alpha$, and for any

bounded set K, we have

$$\|h_L - h\|_K \leqslant L^\alpha \sup_{z\in K} [h]_{\alpha,B(z,L)}. \tag{2.3.6}$$

Finally, we define the local $C^{\alpha-2}$ semi-norm of a distribution ζ for $\alpha - 2 < 0$ as

$$[\zeta]_{\alpha-2,K} = \sup_{L\leqslant 1} \|(\zeta)_L\|_K L^{2-\alpha}. \tag{2.3.7}$$

This is a localised version of the Besov norm of $B^{\alpha-2}_{\infty,\infty}$ as defined, for example in [5, Theorem 2.34]. Any such choice of kernel gives an equivalent norm. Note that, $[\zeta]_{\alpha-2,K}$ depends only on the behaviour of the distribution ζ on the set $K + B(0,1)$ (i.e. if ζ and $\widetilde{\zeta}$ coincide when tested against test-functions supported in this set, then $[\zeta - \widetilde{\zeta}]_{\alpha-2,K} = 0$). Multiplication with a smooth function is a continuous operation with respect to this norm. We have for any smooth and compactly supported function η

$$[\eta\zeta]_{\alpha-2} \leqslant C(\eta)[\zeta]_{\alpha-2,\mathrm{supp}(\eta)}. \tag{2.3.8}$$

Estimates of this type are classical and are typically proved by choosing a convenient mollifying kernel Ψ_L, see e.g. [62] for estimates based on kernels Ψ_L satisfying a semi-group property in L, or [5, Section 2.4] for a proof in the language of Littlewood-Paley theory. We refer to [62, Lemma A3] for a proof that norms defined for different kernels are equivalent. More complicated bounds of this type are also essential in Chapter 3 and are discussed there at length.

We now state our main result, to be proven in Section 2.5.

Theorem 2.3.1. *Assume that $f(u) = u|u|^{m-1} + g(u)$ with $m \geqslant 1$, g bounded and ζ is of regularity $\alpha - 2$ for some $\alpha > 0$ in the sense of* (2.3.7)*. There exists a constant $C = C(\alpha, m, d)$ such that if u is continuous and solves* (2.1.1) *on the cylinder P_0 then for all $R \in (0, \frac{1}{2})$,*

$$\|u\|_{P_R} \leqslant C \max\left\{ R^{-\frac{2}{m-1}}, [\zeta]^{\frac{2}{2+(m-1)\alpha}}_{\alpha-2,P_0}, \|g\|^{\frac{1}{m}} \right\}. \tag{2.3.9}$$

2.4 Maximum principle

We introduce here the essential large-scale ingredient to this whole thesis. In this first chapter, we attempt to make it as general as possible. In the next chapters however we only consider cubic non-linearities and the theorem simplifies greatly in those cases. In particular, the setting of Assumption 2.4.1 is sufficient in that case.

2.4.1 Assumptions and statement

We prove a space-time version of "coming down from infinity" when there is no distribution of negative regularity involved, but we allow for a more general nonlinearity. Let u be a C^2 function defined for $z \in \mathbb{R} \times \mathbb{R}^d$, for which the following holds point-wise for $z \in P_0$ when $u(z) > 0$:

$$(\partial_t - \Delta)u \leqslant -f(u) + g. \tag{2.4.1}$$

Assumption 2.4.1. *We make the following assumptions on f and g:*

1. *g is a bounded function;*
2. *f is C^2 and $f''(u) \geqslant 0$ for $u > 0$;*
3. *there exists a constant $c > 1$ such that $uf'(u) \geqslant cf(u) > 0$ for $u > 0$.*

Define $\Theta(u) = \frac{f(u)}{u}$. By (3), Θ is increasing for $u > 0$.

Theorem 2.4.2. *Let $u \in C^2$ satisfy (2.4.1) for functions f and g satisfying Assumption 2.4.1. There exist $\lambda = \lambda(d) > 0$ and $C = C(c, d)$ such that the following point-wise bound on u holds for all $(t, x) \in (0, 1) \times (-1, 1)^d$:*

$$u(x,t) \leqslant C \max\Big\{\Theta^{-1}\Big(\frac{1}{\lambda^2 \min\{t, (1-x_i)^2, (1+x_i)^2, i = 1...d\}}\Big), f^{-1}(\|g\|)\Big\}. \tag{2.4.2}$$

Note that $\min\{t, (1-x_i)^2, (1+x_i)^2, i = 1...d\}$ is exactly the square of the distance to the boundary of $[0, 1] \times [-1, 1]^d$ in the parabolic metric. Since a similar bound can be obtained for $-u$ under a suitable symmetry assumption, this gives a bound on $||u||_{P_R}$, depending only on R.

The condition $uf'(u) \geqslant cf(u)$ with $c > 1$ is verified exactly for $f(u) = u|u|^{c-1}$, hence any function with at least polynomial growth is included in this theorem. For such monomials, Θ^{-1} becomes $x \mapsto x^{\frac{1}{c-1}}$. For functions with faster growth, the bound is going to be even stronger. However, some functions with super-linear but not polynomial growth are not included. For example $f(u) = u\log(1+u)^\alpha$ for $\alpha > 0$. For this example, $\frac{uf'(u)}{f(u)} = 1 + \frac{u\alpha}{(1+u)\log(1+u)} \to 1$ as $u \to \infty$, so point (3) in Assumption 2.4.1 is violated. We can still get a result in that case, under a slightly weaker, but also slightly more complicated set of assumptions:

Assumption 2.4.3. *We make the following assumptions on f and g:*

1. *g is a bounded function;*
2. *f is C^2 and $uf'(u) \geqslant f(u)$ and there exist two C^2 functions f_1 and f_2 such that $f = f_1 f_2$;*
3. *$f_1'' \geqslant 0$ and $f_1 > 0$ for $u > 0$;*

4. $f_2 \geqslant c > 0$ *for* $u > 0$ *and*

$$f_2(u) \geqslant \max\Big\{ \frac{1}{\Big(\frac{uf_1'(u)}{f_1(u)} - 1\Big)^2}, \frac{1}{\frac{uf_1'(u)}{f_1(u)} - 1} \Big\}. \tag{2.4.3}$$

Define now $\Theta(u) = \frac{f_1(u)}{u}$. Θ *is increasing for* $u > 0$ *by condition* (4).

In the example where we want to take $f_1(u) = u\log(1+u)^\alpha$ for $\alpha > 0$, one can easily check that in order to satisfy condition (4), f_2 should be $\Big(\frac{1+u}{\alpha u}\Big)^2 \frac{\log(1+u)^2}{\alpha}$ and hence $f(u) = \frac{(1+u)^2}{\alpha^2 u}\log(1+u)^{2+\alpha}$ and $\Theta^{-1}(x) = \exp(x^{\frac{1}{\alpha}}) - 1$. Note that this condition is slightly more restrictive than the Osgood condition $\int_1^\infty \frac{1}{f(u)}du < \infty$, which would be satisfied by the function with slightly slower growth $f(u) = u\log(1+u)^{1+\alpha}$ for $\alpha > 0$. Still we believe that our condition may be sharp, and that the slightly stronger growth requirement is due to a genuine difference between the ODE and the PDE setting.

Theorem 2.4.4. *Let* $u \in C^\infty$ *solve* (2.4.1) *for functions* f *and* g *satisfying Assumption 2.4.3. There exist* $\lambda = \lambda(d) > 0$ *and* $C = C(c,d)$ *such that the following point-wise bound on* u*, holds for all* $(t,x) \in (0,1) \times (-1,1)^d$*:*

$$u(x,t) \leqslant C \max\Big\{ \Theta^{-1}\Big(\frac{1}{\lambda^2 \min\{t, (1-x_i)^2, (1+x_i)^2, i = 1...d\}} \Big), f^{-1}(\|g\|) \Big\}. \tag{2.4.4}$$

Theorem 2.4.2 is implied by Theorem 2.4.4 by choosing $f_1 = f$ and $f_2 = \frac{1}{(c-1)^2}$.

Remark 2.4.5. The fact that under these more general assumptions Θ is not simply defined by $f(u)/u$ but instead grows more slowly, is the reason why we do not get an equivalent of Theorem 2.3.1, in the case of slower than polynomial growth.

2.4.2 Bound on the remainder

A first corollary of this result is a "coming down from infinity" result for the singular equation (2.1.1) with general non linearity. In the manner of [22], we expand around the solution to the linear equation: let w solve

$$(\partial_t - \Delta)w = \zeta \qquad \text{on } P_0. \tag{2.4.5}$$

Below in Section 2.6 we construct w as the solution on the whole space of the heat equation with a ζ cutoff outside of P_0, for which $\|w\|$ is bounded by $[\zeta]_{\alpha-2}$, but this particular choice is not essential. Define $v = u - w$. If u is a solution to

$$(\partial_t - \Delta)u(z) = -f(u) + g(u,z) + \zeta,$$

where we assume that f, g satisfies the Assumption 2.4.3 then v is a solution to

$$(\partial_t - \Delta)v(z) = -f(v+w) + g(v+w, z) \tag{2.4.6}$$

on P_0. We now use the w-dependent decomposition $f(v+w) = \widetilde{f}(v,z) + \widetilde{g}(v,z)$ defined by

$$\widetilde{f}(v,z) = \begin{cases} f(v+w) & \text{if } |v(z)| \geqslant 2|w(z)| \\ f\left(\frac{v}{2}\right) & \text{else,} \end{cases}$$

and $\widetilde{g}(v,z) = f(v+w) - \widetilde{f}(v,z)$. Then, on the one hand by monotonicity of f we have $\widetilde{f}(v,z) \geqslant f(\frac{v}{2})$ and on the other hand $\|\widetilde{g}\| \leqslant f(3\|w\|)$. The Assumptions 2.4.3 are then satisfied with $\widetilde{f}$ and $g + \widetilde{g}$ and we can apply Theorem 2.4.4 to get a bound on v, and then the triangle inequality to get bounds on u. We have

$$f^{-1}(\|g + \widetilde{g}\|) \leqslant f^{-1}(2\|g\|) + 6\|w\|.$$

A corollary of Theorem 2.4.4 is then:

Corollary 2.4.6. *Assume $\zeta \in C^{\alpha-2}$ for some $\alpha > 0$. If u is solution to* (2.1.1) *and w is solution to* (2.4.5)*, then there exists constants $C = C(c, d, \alpha)$ and $\lambda = \lambda(d)$ such that*

$$\|u\|_{P_R} \leqslant C \max\Big\{\Theta^{-1}((\lambda R)^{-2}), f^{-1}(2\|g\|), \|w\|\Big\}. \tag{2.4.7}$$

Keeping in mind the motivation of stochastic PDEs, where ζ is the white noise, the drawback of the expansion around the solution to the linear equation is that the integrability of u that we get out of this result is at best the one of w. As we will see in Section 2.7, Theorem 2.4.4 allows for better estimates than this in the polynomial case.

2.4.3 Proof of Theorem 2.4.4

We split the proof of the theorem in two lemmas. Lemma 2.4.7 states conditions on a function η that imply a bound on the product $u\eta$, and Lemma 2.4.8 gives a particular choice of η that satisfies the conditions of Lemma 2.4.7 and implies Theorem 2.4.4.

Lemma 2.4.7. *Let η be a continuous function defined on $\mathbb{R}_+ \times [-1,1]^d$, C^2 and strictly positive on the interior and such that $\eta = 0$ on the boundary. Assume that η satisfies the following inequalities:*

$$\frac{(\partial_t - \Delta)\eta}{\eta} + 2\frac{|\nabla\eta|^2}{\eta^2} \leqslant \frac{\eta}{2} f(\frac{1}{\eta}), \tag{2.4.8}$$

and

$$0 \leqslant \eta \leqslant \frac{1}{f^{-1}(\|g\|)}. \tag{2.4.9}$$

Then if u solves (2.4.1) *it satisfies the bound $u\eta \leqslant 2$.*

Proof. Take u satisfying (2.4.1). Either $u\eta$ attains its maximum on $[0,1] \times [-1,1]^d$ at some point $z_0 \in (0,1] \times (-1,1)^d$, or it is non-positive, in which case $u \leqslant 0$ in $[0,1] \times \{|x| \leqslant 1\}$. Assuming this is not the case, we get that at the maximum point, $0 = \nabla(u\eta)(z_0)$, i.e.

$$\nabla u = -\frac{\nabla \eta}{\eta} u. \tag{2.4.10}$$

If $z_0 \in \{1\} \times (-1,1)^d$, then $\partial_t \eta u(z_0) \geqslant 0$. Else, $\partial_t \eta u(z_0) = 0$. Additionally, $\Delta \eta u(z_0) \leqslant 0$ and therefore at the maximum we have

$$\begin{aligned} 0 \leqslant & (\partial_t - \Delta)(u\eta) = \eta(\partial_t - \Delta)u + u(\partial_t - \Delta)\eta - 2\nabla u \cdot \nabla \eta \\ \overset{(2.4.1);(2.4.10)}{\leqslant} & -\eta(f(u) - g) + u\Big((\partial_t - \Delta)\eta + 2\frac{|\nabla \eta|^2}{\eta}\Big). \end{aligned}$$

Then we get from (2.4.8)

$$\frac{f(u)}{u} \leqslant \frac{\eta}{2} f\Big(\frac{1}{\eta}\Big) + \frac{\|g\|}{u} \leqslant 2 \max\Big\{\frac{\eta}{2} f\Big(\frac{1}{\eta}\Big), \frac{\|g\|}{u}\Big\}. \tag{2.4.11}$$

If the maximum is realised by the first term, then $\frac{f(u)}{u} \leqslant \eta f(\frac{1}{\eta})$. Since $uf'(u) \geqslant f(u)$, $u \mapsto \frac{f(u)}{u}$ is increasing, we have that at z_0, $u\eta \leqslant 1$. If the maximum is realised by the second term, then it has to be bigger than the first one :

$$\frac{\eta}{2} f\Big(\frac{1}{\eta}\Big) \leqslant \frac{\|g\|}{u} \Rightarrow u\eta \leqslant 2\frac{\|g\|}{f(\frac{1}{\eta})}.$$

We then have that at z_0, $u\eta \leqslant 2$ under the condition (2.4.9). □

Thus proving a bound on u reduces to choosing a suitable function η satisfying the inequalities (2.4.8) and (2.4.9). Ideally, we would like to take η directly as

$$\eta(x,t) = \min\Big\{\Theta^{-1}\Big(\frac{1}{\lambda^2 t}\Big)^{-1}, \Theta^{-1}\Big(\frac{1}{\lambda^2(1-x_i)^2}\Big)^{-1}, i = 1...d, f^{-1}(\|g\|)^{-1}\Big\}.$$

This *almost* works as the individual function within the $\min$ *do* satisfy (2.4.8). Indeed, for example $\eta(x,t) = \Theta^{-1}\Big(\frac{1}{\lambda^2 t}\Big)^{-1}$ satisfies

$$\partial_t \eta = \frac{\lambda^2}{(\lambda^2 t)^2} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 t})} \eta^2. \tag{2.4.12}$$

We use $v = \Theta^{-1}(\frac{1}{\lambda^2 t}) \leqslant \frac{1}{\eta}$ and $f = f_1 f_2$. Given that $\Theta'(y) = \frac{f_1'(y)}{y} - \frac{f_1(y)}{y^2}$, we get

$$\frac{\partial_t \eta}{\eta^2 f(\frac{1}{\eta})} \leqslant \frac{\lambda^2}{f(\frac{1}{\eta})} \frac{\Theta(v)^2}{\Theta'(v)} \leqslant \frac{\lambda^2}{f_1(v) f_2(v)} \frac{\Theta(v)^2}{\Theta'(v)} = \frac{\lambda^2}{f_2(v)} \frac{1}{\frac{v f_1'(v)}{f_1(v)} - 1}.$$

Applying the condition (2.4.3) gives a bound on this, independent of v, establishing (2.4.8). Unfortunately, taking the $\min$ of these functions is *not* an admitted operation due to the discontinuity of the derivative at the points where we join two solutions (recall that while the minimum of two supersolutions is a supersolution the maximum of two supersolutions in general is not).

In the following lemma we overcome this problem by replacing the non-smooth function $\min\{x_1, \dots, x_n\}$ in the above definition of η by the smooth function

$$\frac{1}{x_1^{-1} + \dots x_n^{-1}}.$$

Lemma 2.4.8. *For $z = (t, x) \in (0, \infty) \times (-1, 1)$, we define*

$$\eta(x,t) = \frac{1}{\Theta^{-1}(\frac{1}{\lambda^2 t}) + \sum_{i=1}^d \Big(\Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2}) + \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2}) \Big) + f^{-1}(\|g\|)}, \tag{2.4.13}$$

where $\lambda = (28d+1)^{-\frac{1}{2}}$ and we continuously extend with the value 0 on the boundary of the domain. Then η satisfies (2.4.8) *and* (2.4.9) *and*

$$\frac{1}{\eta} \leqslant (2d+1)\Theta^{-1}\Big(\frac{1}{\lambda^2 \min_i\{t, (1+x_i)^2, (1-x_i)^2\}}\Big) + f^{-1}(\|g\|) \tag{2.4.14}$$

This choice of η guarantees a bound on u that is related to the distance from the boundary of $[0,1] \times [-1,1]^d$, independently of the boundary conditions. Lemmas 2.4.7 and 2.4.8 imply directly Theorem 2.4.4.

Proof. The condition (2.4.9) is obvious, and so is (2.4.14). We have a bit more:

$$\begin{aligned}(2d+1)\Theta^{-1}\Big(\frac{1}{\lambda^2 \min_i\{t, (1+x_i)^2, (1-x_i)^2\}}\Big) \\ \geqslant \frac{1}{\eta} - f^{-1}(\|g\|) \geqslant \Theta^{-1}\Big(\frac{1}{\lambda^2 \min_i\{t, (1+x_i)^2, (1-x_i)^2\}}\Big).\end{aligned} \tag{2.4.15}$$

We will now check (2.4.8). For the time variable, we have as in (2.4.12)

$$\partial_t \eta = \frac{\lambda^2}{(\lambda^2 t)^2} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 t})} \eta^2,$$

and we have already seen that this implies

$$\frac{\partial_t \eta}{\eta^2 f(\frac{1}{\eta})} \leqslant \lambda^2.$$

We now consider the spatial derivatives.

$$\partial_i \eta = \frac{1}{\lambda^2} \Big(\frac{2}{(1+x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2})} - \frac{2}{(1-x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2})} \Big) \eta^2,$$

$$\begin{aligned}
\partial_i^2 \eta = &- \frac{1}{\lambda^2} \Big(\frac{6}{(1+x_i)^4} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2})} + \frac{6}{(1-x_i)^4} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2})} \Big) \eta^2 \\
&+ \frac{1}{\lambda^4} \Big(\frac{4}{(1+x_i)^6} \frac{\Theta'' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2})}{\big(\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2})\big)^3} + \frac{4}{(1-x_i)^6} \frac{\Theta'' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2})}{\big(\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2})\big)^3} \Big) \eta^2 \\
&+ \frac{2}{\lambda^4} \Big(\frac{2}{(1+x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2})} - \frac{2}{(1-x_i)^3} \frac{1}{\Theta' \circ \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2})} \Big)^2 \eta^3.
\end{aligned}$$

Note that the last line is equal to $\frac{\partial_i \eta}{\eta^2} 2\eta \partial_i \eta = 2\frac{(\partial_i \eta)^2}{\eta}$, hence it will cancel when computing $-\partial_i^2 \eta + 2\frac{(\partial_i \eta)^2}{\eta}$. For the remaining terms, we use $v_{1,i} = \Theta^{-1}(\frac{1}{\lambda^2 (1+x_i)^2})$ and $v_{2,i} = \Theta^{-1}(\frac{1}{\lambda^2 (1-x_i)^2})$ and we get:

$$\begin{aligned}
\frac{1}{\lambda^2 f(\frac{1}{\eta})} \Big(- \frac{\partial_i^2 \eta}{\eta^2} + 2\frac{(\partial_i \eta)^2}{\eta^3} \Big) = &\frac{6\Theta(v_{1,i})^2}{f(\frac{1}{\eta})(\frac{f_1'(v_{1,i})}{v_{1,i}} - \frac{f_1(v_{1,i})}{v_{1,i}^2})} + \frac{6\Theta(v_{2,i})^2}{f(\frac{1}{\eta})(\frac{f_1'(v_{2,i})}{v_{2,i}} - \frac{f_1(v_{2,i})}{v_{2,i}^2})} \\
&- \frac{4\Theta(v_{1,i})^3}{f(\frac{1}{\eta})} \frac{\frac{f''(v_{1,i})}{v_{1,i}} - \frac{2}{v_{1,i}}(\frac{f_1'(v_{1,i})}{v_{1,i}} - \frac{f_1(v_{1,i})}{v_{1,i}^2})}{(\frac{f_1'(v_{1,i})}{v_{1,i}} - \frac{f_1(v_{1,i})}{v_{1,i}^2})^3} \\
&- \frac{4\Theta(v_{2,i})^3}{f(\frac{1}{\eta})} \frac{\frac{f''(v_{2,i})}{v_{2,i}} - \frac{2}{v_{2,i}}(\frac{f_1'(v_{2,i})}{v_{2,i}} - \frac{f_1(v_{2,i})}{v_{2,i}^2})}{(\frac{f_1'(v_{2,i})}{v_{2,i}} - \frac{f_1(v_{2,i})}{v_{2,i}^2})^3}.
\end{aligned}$$

Using that f is increasing, the bound (2.4.15) and $f = f_1 f_2$, we have that $f(\frac{1}{\eta}) \geqslant f_1(v_{j,i}) f_2(v_{j,i})$ for $j \in \{1,2\}$. We also know that $f_2'' > 0$, hence we get:

$$\frac{1}{f(\frac{1}{\eta})} \Big(- \frac{\partial_i^2 \eta}{\eta^2} + 2\frac{(\partial_i \eta)^2}{\eta^3} \Big) \leqslant \frac{6\lambda^2}{f_2(v_{1,i})(\frac{f_1'(v_{1,i}) v_{1,i}}{f_1(v_{1,i})} - 1)} + \frac{6\lambda^2}{f_2(v_{2,i})(\frac{f_1'(v_{2,i}) v_{2,i}}{f_1(v_{2,i})} - 1)}$$

$$+\frac{8\lambda^2}{f_2(v_{1,i})(\frac{f_1'(v_{1,i})v_{1,i}}{f_1(v_{1,i})}-1)^2}+\frac{8\lambda^2}{f_2(v_{2,i})(\frac{f_1'(v_{2,i})v_{2,i}}{f_1(v_{2,i})}-1)^2}.$$

We conclude this proof by using the condition (2.4.3) and the value $\lambda=(28d+1)^{-\frac{1}{2}}$. $\square$

2.5 Proof of Theorem 2.3.1

From now on, $f(u)=|u|^{m-1}u$. In particular, Theorem 2.4.2 holds with $\Theta^{-1}(R^{-2})=R^{-\frac{1}{m-1}}$. The proof relies on two arguments. The small scale oscillations are controlled via Schauder theory and the large scale behaviour through the maximum principle derived in Section 2.4, which applies only to regular objects. A connection between the two is established via the convolution of the equation with the kernel introduced in Section 2.3, which produces a commutator term. The technicality of the proof lies in balancing the contribution of the commutator and the contribution of the irregular noise.

Throughout the proof, $\lesssim$ will denote a bound up to a multiplicative constant which may change from line to line, but will only depend on d, m and α. We will also write u^m as a short-hand for $u|u|^{m-1}$, as in the case when m is an odd integer.

STEP 1: Local Schauder estimate We claim that for any $R>0$, for any $k>2$,

$$\begin{aligned}[u]_{\alpha,B(z,R)}&\lesssim\sup_{L\leqslant kR}L^{2-\alpha}\|(\mathbf{1}_{\{B(z,kR)\}}(\partial_t-\Delta)u)_L\|+(kR)^{-\alpha}\|u\|_{B(z,kR)}\\&\leqslant(kR)^{2-\alpha}\|u\|^m_{B(z,kR)}+(kR)^{2-\alpha}\|g\|+[\zeta]_{\alpha-2,B(z,kR)}\\&\quad+(kR)^{-\alpha}\|u\|_{B(z,kR)}.\end{aligned}\tag{2.5.1}$$

We prove this estimate by applying a cutoff function and using the low-regularity Schauder estimate given in Lemma 2.9.1. By scaling and translation, it is enough to prove for some C^α function U,

$$[U]_{\alpha,B(0,\frac{1}{2})}\lesssim\sup_{L\leqslant 1}L^{2-\alpha}\|(\mathbf{1}_{\{B(0,1)\}}(\partial_t-\Delta)U)_L\|+\|U\|_{B(0,1)}.\tag{2.5.2}$$

Indeed, since $[u]_{\alpha,B(0,\frac{1}{k})}\leqslant[u]_{\alpha,B(0,\frac{1}{2})}$, if we have (2.5.2), define

$$U(t,x)=u((kR)^2(t-t_0),kR(x-x_0)).$$

Then

$$\|u\|_{B(z,kR)}=\|U\|_{B(0,1)},\quad [u]_{\alpha,B(z,R)}=(kR)^\alpha[U]_{\alpha,B(0,\frac{1}{k})}$$

$$\sup_{L\leqslant kR}L^{2-\alpha}\|(\mathbf{1}_{\{B(z,kR)\}}(\partial_t-\Delta)u)_L\|=\sup_{L\leqslant 1}L^{2-\alpha}\|(\partial_t-\Delta)(U\mathbf{1}_{\{B(0,1)\}})_L\|.$$

We proceed to prove (2.5.2). Let η be a cutoff function, with value 1 on $B(0,\frac{1}{2})$ and 0 on $B(0,1)^C$, and such that $\|\nabla\eta\|$ and $\|(\Delta+\partial_t)\eta\|$ are bounded by an independent constant. Then

$$(\partial_t-\Delta)U\eta=\eta(\partial_t-\Delta)U+U(\partial_t+\Delta)\eta-2\nabla.(U\nabla\eta). \tag{2.5.3}$$

By applying our version of the Schauder estimate, Lemma 2.9.1 to Equation (2.5.3) we get that:

$$[U\eta]_\alpha\lesssim\sup_{0<L<1}L^{2-\alpha}\|(\eta(\partial_t-\Delta)U+U(\partial_t+\Delta)\eta-2\nabla.(U\nabla\eta))_L\|. \tag{2.5.4}$$

We apply the triangle inequality and make use of (2.3.8) to bound each of these terms as follows.

$$\|(\eta(\partial_t-\Delta)U)_L\|\lesssim\|(\mathbf{1}_{\{B(0,1)\}}(\partial_t-\Delta)U)_L\|,$$

$$\|(U(\partial_t+\Delta)\eta)_L\|\leqslant\|U(\partial_t+\Delta)\eta\|\lesssim\|U\|_{B(0,1)},$$

$$\begin{aligned}\|(\nabla.(U\nabla\eta))_L\|=&\sup_z\int(U\nabla\eta)(z-\overline{z}).\nabla\Psi_L(\overline{z})d\overline{z}\\ \leqslant&\|U\nabla\eta\|\|\nabla\Psi_L\|_{L^1}\\ \lesssim&\frac{1}{L}\|U\|_{B(0,1)}.\end{aligned}$$

Since $\alpha<1$, we have

$$\begin{aligned}\sup_{0<L<1}&L^{2-\alpha}\|(\eta(\partial_t-\Delta)U+U(\partial_t+\Delta)\eta-2\nabla.(U\nabla\eta))_L\|\\ &\lesssim\|(\mathbf{1}_{\{B(0,1)\}}(\partial_t-\Delta)U)_L\|+\|U\|_{B(0,1)}.\end{aligned}$$

This concludes the proof of (2.5.2), hence the proof of (2.5.1).

STEP 2: Application of the maximum principle We convolve the equation (2.1.1) with Ψ_L, where $L\in(0,1)$ will be specified later:

$$(\partial_t-\Delta)u_L=-(u_L)^m+g_L+\zeta_L+\left((u_L)^m-(u^m)_L\right). \tag{2.5.5}$$

The anti-symmetry of $u\mapsto u^m$ and Theorem 2.4.4 implies that for all $0<R'<R<\frac{1}{2}$,

$$\|u_L\|_{P_R} \lesssim \max\Big\{ \left(\frac{1}{(R-R')^2}\right)^{\frac{1}{m-1}}, \|g\|^{\frac{1}{m}}, \|\zeta_L\|^{\frac{1}{m}}_{P_{R'}}, \big(\|(u_L)^m - (u^m)_L\|_{P_{R'}}\big)^{\frac{1}{m}} \Big\}. \tag{2.5.6}$$

The goal is now to balance the commutator and the term with the noise. This will be done by choosing the parameter L appropriately in Step 5 below.

STEP 3: Bounds on the commutator We need estimates on the commutator $(u_L)^m - (u^m)_L$. This is obtained as u is C^α, using the moment bounds (2.3.5) and (2.3.6).

$$\begin{aligned}
((u_L)^m - (u^m)_L)(z) &= \int \Psi_L(z-\overline{z})\,(u_L(z)^m - u(\overline{z})^m)\,d\overline{z} \\
&= \int \Psi_L(z-\overline{z}) \int_0^1 ((u)_L(z) - u(\overline{z}))\, m\,(\lambda(u)_L(z) + (1-\lambda)u(\overline{z}))^{m-1}\,d\lambda d\overline{z} \\
&\leqslant m\|u\|^{m-1}_{B(z,L)} \int \Psi_L(z-\overline{z})\,(u_L(z) - u(z) + u(z) - u(\overline{z}))\,d\overline{z} \\
&\leqslant m\|u\|^{m-1}_{B(z,L)} \int \Psi_L(z-\overline{z})\,\left(L^\alpha [u]_{\alpha,B(z,L)} + [u]_{\alpha,B(z,L)} d(z,\overline{z})^\alpha\right) d\overline{z} \\
&\leqslant 2m\|u\|^{m-1}_{B(z,L)} L^\alpha [u]_{\alpha,B(z,L)}.
\end{aligned}$$

Since this is true for all $z \in P_R$,

$$\|(u_L)^m - (u^m)_L\|_{P_R} \leqslant 2m\|u\|^{m-1}_{P_{R-L}} \sup_{z\in P_R} [u]_{\alpha,B(z,L)} L^\alpha. \tag{2.5.7}$$

Using the local Schauder estimate (2.5.1) gives, for any $k > 2$:

$$\begin{aligned}
\|(u_L)^m - (u^m)_L\|_{P_R} \lesssim\ & L^2 k^{2-\alpha}(\|u\|^{2m-1}_{P_{R-kL}} + \|u\|^{m-1}_{P_{R-kL}}\|g\|) \\
& + \|u\|^{m-1}_{P_{R-L}} \sup_{z\in P_R} [\zeta]_{\alpha-2,B(z,kR)} L^\alpha + k^{-\alpha}\|u\|^m_{P_{R-kL}}.
\end{aligned} \tag{2.5.8}$$

STEP 4: Boot-strapping We show here that for k, L such that $2(k+1)L \leqslant 1$, with as before $k > 2$, and for $1 \geqslant R \geqslant 2(k+1)L$, we have

$$\begin{aligned}
\|u\|_{P_R} \lesssim \max\Big\{ & R^{\frac{2}{1-m}}, \|g\|^{\frac{1}{m}}, \left([\zeta]_{\alpha-2,P_0} L^{\alpha-2}\right)^{\frac{1}{m}}, (L^2 k^{2-\alpha})^{\frac{1}{m}} \|u\|^{2-\frac{1}{m}}_{P_0}, \\
& \left(L^2 k^{2-\alpha}\right)^{\frac{1}{m}} \|u\|^{1-\frac{1}{m}}_{P_0} \|g\|^{\frac{1}{m}}, \left(\|u\|^{m-1}_{P_0} [\zeta]_{\alpha-2,P_0} L^\alpha\right)^{\frac{1}{m}}, k^{-\frac{\alpha}{m}} \|u\|_{P_0}, \\
& L^2 k^{2-\alpha}\|u\|^m_{P_0}, L^2 k^{2-\alpha}\|g\|, L^\alpha [\zeta]_{\alpha-2,P_0}, k^{-\alpha}\|u\|_{P_0} \Big\}.
\end{aligned} \tag{2.5.9}$$

We need to be careful with the sets over which the supremum norms are taken, since our different estimates always require a bit more space. We use the bound (2.3.6) and the Schauder estimate (2.5.1) with L playing the role of R:

$$\begin{aligned}\|u\|_{P_R} \leqslant & \|u_L\|_{P_R} + L^\alpha \sup_{z\in P_R} [u]_{\alpha,B(z,L)} \\ \leqslant & \|u_L\|_{P_{R-L}} + L^2k^{2-\alpha}(\|u\|^m_{P_{R-kL}} + \|g\|) \\ & + L^\alpha[\zeta]_{\alpha-2,P_{R-kL}} + k^{-\alpha}\|u\|_{P_{R-kL}}.\end{aligned} \tag{2.5.10}$$

Defining $r = kL$ permits to apply the bounds (2.5.6), (2.5.8) and (2.3.7). As $r \geqslant 0$, $P_r \subset P_0$.

$$\begin{aligned}\|u_L\|_{P_{R-L}} \lesssim & \max\Big\{(R-(k+1)L)^{\frac{2}{1-m}}, \|g\|^{\frac{1}{m}}, \|\zeta_L\|^{\frac{1}{m}}_{P_r}, \|(u_L)^m - (u^m)_L\|^{\frac{1}{m}}_{P_r}\Big\} \\ \lesssim & \max\Big\{(R-(k+1)L)^{\frac{2}{1-m}}, \|g\|^{\frac{1}{m}}, \left([\zeta]_{\alpha-2,P_0}L^{\alpha-2}\right)^{\frac{1}{m}}, (L^2k^{2-\alpha})^{\frac{1}{m}}\|u\|^{2-\frac{1}{m}}_{P_0}, \\ & \left(L^2k^{2-\alpha}\right)^{\frac{1}{m}}\|u\|^{1-\frac{1}{m}}_{P_0}\|g\|^{\frac{1}{m}}, \left(\|u\|^{m-1}_{P_0}[\zeta]_{\alpha-2,P_0}L^\alpha\right)^{\frac{1}{m}}, k^{-\frac{\alpha}{m}}\|u\|_{P_0}\Big\}.\end{aligned} \tag{2.5.11}$$

If we start with $R \geqslant 2(k+1)L$ then $R-(k+1)L \geqslant \frac{R}{2}$. Putting together (2.5.11) and (2.5.10) gives (2.5.9).

STEP 5: Choosing L In order to balance the term containing ζ in (2.5.9), we see that we should assign the value $L = \frac{\mu}{\|u\|^{\frac{m-1}{2}}_{P_0}}$ for some $\mu \in (0,1)$ to be chosen. Note also that as $\mu \in (0,1)$, $(\mu^{\alpha-2} \vee \mu^\alpha) = \mu^{\alpha-2}$. Furthermore, we impose $\mu^2k^{2-\alpha} \leqslant 1$. Consequently, (2.5.9) becomes

$$\begin{aligned}\|u\|_{P_R} \lesssim \max\Big\{ & R^{\frac{2}{1-m}}, (1+(\mu^2k^{2-\alpha})^{\frac{1}{m}})\|g\|^{\frac{1}{m}}, \left(\mu^{\alpha-2}\|u\|^{(m-1)\frac{2-\alpha}{2}}_{P_0}[\zeta]_{\alpha-2,P_0}\right)^{\frac{1}{m}}, \\ & ((\mu^2k^{2-\alpha}\vee k^{-\alpha})^{\frac{1}{m}} + (\mu^2k^{2-\alpha}\vee k^{-\alpha}))\|u\|_{P_0}, \\ & \mu^2k^{2-\alpha}\|u\|^{1-m}_{P_0}\|g\|, \frac{\mu^\alpha}{\|u\|^{(m-1)\frac{\alpha}{2}}_{P_0}}[\zeta]_{\alpha-2,P_0}\Big\}.\end{aligned} \tag{2.5.12}$$

STEP 6: Identification of terms We claim that the bound above implies that there exists a positive constants C such that:

$$\|u\|_{P_R} \leqslant C\max\Big\{R^{\frac{2}{1-m}}, \|g\|^{\frac{1}{m}}, [\zeta]^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}_{\alpha-2,P_0}, \frac{1}{2C}\|u\|_{P_0}\Big\}. \tag{2.5.13}$$

We need to interpolate some of the arguments of the maximum in (2.5.12) with arguments of our goal (2.5.13). The first two terms are already in the right form. For the next one, Young's inequality gives that for any $\gamma > 0$,

$$\left(\mu^{\alpha-2}\|u\|_{P_0}^{(m-1)\frac{2-\alpha}{2}}[\zeta]_{\alpha-2,P_0}\right)^{\frac{1}{m}} \lesssim \mu^{\frac{\alpha-2}{m}} \max\left\{\gamma\|u\|_{P_0}, \gamma^{\frac{\alpha-2}{\frac{2}{m-1}+\alpha}}[\zeta]_{\alpha-2,P_0}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}\right\}.$$

The next term is also in the right form, provided one chooses first k large, and then μ small, both independently of $\|u\|_{P_0}$ and satisfying all conditions above. The only condition that was not independent of $\|u\|_{P_0}$ was $1 \geqslant 2(k+1)L = 2(k+1)(\mu\|u\|_{P_0}^{-1})^{(m-1)/2}$. However, if this condition is not satisfied after choosing k then μ according to all other conditions above, it means that we have a bound on $\|u\|_{P_0}$:

$$\|u\|_{P_0} < (2(k+1)\mu)^{\frac{2}{m-1}},$$

which proves the theorem directly.

The last two terms can not be dealt with classical interpolation, since they involve negative powers of $\|u\|_{P_0}$. For the first one, we state that always one of the following is true, for any $\gamma > 0$:

$$\|u\|_{P_0}^{-(m-1)}\|g\| \leqslant \gamma\|u\|_{P_0} \text{ or } \|u\|_{P_0}^{m} \leqslant \frac{1}{\gamma}\|g\|.$$

The first case gives the last argument of our objective for γ small enough. The second case gives $\|u\|_{P_R} \leqslant \|u\|_{P_0} \leqslant (\frac{1}{\gamma}\|g\|)^{\frac{1}{m}}$. We proceed similarly for the last term. One of the following is always true:

$$\mu^{\alpha}\|u\|_{P_0}^{-(m-1)\frac{\alpha}{2}}[\zeta]_{\alpha-2,P_0} \leqslant \mu^{\alpha}\gamma\|u\|_{P_0} \text{ or } \|u\|_{P_0}^{1+(m-1)\frac{\alpha}{2}} \leqslant \frac{1}{\gamma}[\zeta]_{\alpha-2,P_0}.$$

Once again the first case gives the last argument of our objective for $\mu^{\alpha}\gamma$ small enough, and the second case gives $\|u\|_{P_R} \leqslant \|u\|_{P_0} \leqslant (\frac{1}{\gamma}[\zeta]_{\alpha-2,P_0})^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}$. We can then choose k large, μ and γ small to get the desired constant C.

STEP 7: Iterating the result The last argument of the maximum (2.5.13) is greater than the first one for all R such that

$$R \leqslant R_1 := \left(\frac{1}{2C}\|u\|_{P_0}\right)^{\frac{1-m}{2}},$$

Let us check that this is not in contradiction with $R_1 \geqslant 2(k+1)L$. By defintion of L

and R_1,

$$2(k+1)L = 2(k+1)\frac{\mu}{\|u\|_{P_0}^{\frac{m-1}{2}}} = 2(k+1)\mu(2C)^{\frac{1-m}{2}}R_1 \leqslant R_1 \Leftrightarrow 2(k+1)\mu(2C)^{\frac{1-m}{2}} \leqslant 1.$$

Since $C > 1$ and $m > 1$, it is enough to have $2(k+1)\mu \leqslant 1$. This can be done since μ is chosen after k.

From this point, the result (2.5.13) can be iterated to get bounds for smaller and smaller parabolic boxes.

$$\|u\|_{P_{(R+R_{n-1})}} \leqslant C \max\left\{\frac{1}{2C}\|u\|_{P_{R_{n-1}}}, \|g\|^{\frac{1}{m}}, [\zeta]_{\alpha-2,P_0}^{\frac{1}{1+(m-1)\frac{\alpha}{2}}}, \frac{1}{R^{\frac{2}{m-1}}}\right\}.$$

Define R_n recursively by

$$R_n - R_{n-1} = \left(\frac{1}{2C}\|u\|_{P_{R_{n-1}}}\right)^{\frac{1-m}{2}} = \left(\frac{1}{2C}\frac{\|u\|_{P_0}}{2^{n-1}}\right)^{\frac{1-m}{2}}. \tag{2.5.14}$$

We conclude by summing those increments:

$$\begin{aligned} R_n &= \sum_{k=1}^{n} R_k - R_{k-1} = \sum_{k=1}^{n}\left(\frac{1}{2C}\frac{\|u\|_{P_0}}{2^{k-1}}\right)^{\frac{1-m}{2}}. \\ &= \left(\frac{\|u\|_{P_0}}{2C}\right)^{\frac{1-m}{2}} \sum_{k=0}^{n-1}(2^{\frac{1-m}{2}})^k \lesssim \left(\frac{\|u\|_{P_0}}{2C}\right)^{\frac{1-m}{2}} \end{aligned} \tag{2.5.15}$$

The same arguments as in the proof of Lemma 2.2.1 concludes the proof of Theorem 2.3.1.

2.6 Mutiplicative noise

We present an example of equation where our result applies. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and let $(W(t,\eta), t \geqslant 0, \eta \in C_0^\infty(\mathbb{R}^d))$ be a Brownian motion with spatial covariance operator K on Ω. We assume that K is given by the convolution with a function with controlled blow-up near the origin, i.e.

$$K\phi(x) = \int_{\mathbb{R}^d} K(x-x')\phi(x')dx', \tag{2.6.1}$$

for $K \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfying

$$|K(x)| \leqslant \frac{1}{|x|^\lambda}, \tag{2.6.2}$$

for some $\lambda < 2$. If $\lambda > 1$ and $d = 1$, we allow additionally for a Dirac mass in the origin, in which case (2.6.1) turns into

$$K\phi(x) = \int_{\mathbb{R}} K(x - x')\phi(x')dx' + \phi(x). \tag{2.6.3}$$

In other words $(W(t, \eta), t \geqslant 0, \eta \in C_0^\infty(\mathbb{R}^d))$ is a centred Gaussian process with covariances given either by

$$\mathbb{E}W(t, \phi)W(t', \phi') = (t \wedge t') \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x)K(x - x')\phi'(x')dxdx'$$

or in the one-dimensional case

$$\begin{aligned}&\mathbb{E}W(t, \phi)W(t', \phi') \\ &\quad = (t \wedge t')\Big[\int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x)K(x - x')\phi'(x')dxdx' + \int_{\mathbb{R}} \phi(x)\phi'(x)dx\Big].\end{aligned}$$

Let $(\sigma(t, x), t \geqslant 0, x \in \mathbb{R}^d)$ be a progressively measurable process, with a deterministic L^∞ bound, without loss of generality $|\sigma(t, x)| \leqslant 1$. Let $u(t, x)$ be a continuous process which satisfies the SPDE

$$du = (\Delta u - f(u) + g(u))dt + \sigma dW \tag{2.6.4}$$

on P_0, with f satisfying the Assumptions 2.4.3. More precisely, for all $\eta \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ compactly supported in P_0 we assume that the following holds almost surely:

$$\begin{aligned}&\int \int u(-\partial_t - \Delta)\eta dtdx \\ &\quad = \int \int (-f(u, z) + g(u, z))\eta dtdx + \int \int \eta(x)\sigma(t, x)dxdW(t, x),\end{aligned} \tag{2.6.5}$$

where $\int \int \eta(x)\sigma(t, x)dxdW(t, x)$ should be interpreted as a stochastic integral, as defined in [24, Chapter 4]. The following lemma shows that the results of our deterministic analysis are applicable to this stochastic case.

The Theorem 2.3.1 does not depend on the particular choice of convolution kernel Ψ, and different choices of kernel don't change the Hölder norms. We apply it with $\widetilde{\Psi}$ defined as

$$\widetilde{\Psi} = \Psi_{\frac{1}{2}} * \Psi_{\frac{1}{2}}, \tag{2.6.6}$$

where Ψ is as defined in Section 2.3. It is clear that $\widetilde{\Psi}$ is still non-negative, smooth and compactly supported in $B(0, 1)$. We still write $(\cdot)_L$ for the convolution with Ψ_L but we

define the $C^{\alpha-2}$ norm with respect to $\widetilde{\Psi}$

$$[\zeta]_{\alpha-2,C} = \sup_{0<L\leqslant 1} \|(\zeta_{\frac{L}{2}})_{\frac{L}{2}}\|_C L^{2-\alpha}$$

Lemma 2.6.1. *We define a family of random variables* $(\zeta(\eta), \eta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d))$ *by*

$$\zeta(\eta) = \int\int \eta(x)\sigma(t,x)dxdW(t,x).$$

Then there exists a random distribution $\widetilde{\zeta}$ *on* Ω *which almost surely takes values in* $C^{\alpha-2}$ *for any* $\alpha < \frac{2-\lambda}{2}$ *and such that for* $\varepsilon > 0$ *small enough*

$$\mathbb{E}\left[\exp\left(\epsilon[\widetilde{\zeta}]^2_{\alpha-2,P_0}\right)\right] < \infty. \tag{2.6.7}$$

Furthermore $\widetilde{\zeta}$ *is a modification of* ζ *in the sense that for all* $\eta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d)$ *we have almost surely*

$$\widetilde{\zeta}(\eta) = \zeta(\eta).$$

We have the following corollaries.

Corollary 2.6.2. *Let* u *solve the SPDE* (2.6.4) *in the sense of* (2.6.5) *for* f *and* g *satisfying Assumption 2.4.3. Define* $\Theta(u) = \frac{f_1(u)}{u}$. *Then there exists* $\varepsilon_0 = \varepsilon_0(c,d,\alpha) > 0$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0$,

$$\mathbb{E}\left[\exp\left(\varepsilon\Big(\sup_{0<R\leqslant\frac{1}{2}} \frac{\|u\|_{P_R}}{\Theta^{-1}((\lambda R)^{-2})}\Big)^2\right)\right] < \infty.$$

Proof. Let ρ be a cutoff outside P_0 supported on $P_0 + B(0,1)$, and w be the bounded solution to

$$(\partial_t - \Delta)w = \rho\xi,$$

vanishing for $t < -2$. From Lemma 2.9.2, $\|w\| \lesssim [\xi]_{\alpha-2}$. Using Corollary 2.4.6 and Lemma 2.6.1 finishes the proof. □

Using Theorem 2.3.1 in the case $f(u) = u|u|^{m-1}$, we have the following improved estimate:

Corollary 2.6.3. *Let* u *solve the SPDE* (2.6.4) *in the sense of* (2.6.5) *where* $f(u,z) = u|u|^{m-1}$ *and* g *is bounded. Then there exists* $\varepsilon_0 = \varepsilon_0(m,d,\alpha) > 0$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0$,

$$\mathbb{E}\left[\exp\left(\varepsilon\Big(\sup_{0<R\leqslant\frac{1}{2}} R^{\frac{2}{m-1}}\|u\|_{P_R}\Big)^{2+(m-1)\alpha}\right)\right] < \infty.$$

The proof of Lemma 2.6.1 relies on the following technical lemma.

Lemma 2.6.4. *The supremum* $\sup_{0<L\leqslant 1} \|\zeta_L\|_{P_0}^{2p} L^{2p(2-\alpha)}$ *is bounded by the supremum over dyadic* L *only,*

$$\sup_{0<L\leqslant 1} \|\zeta_L\|_{P_0}^{2p} L^{2p(2-\alpha)} \lesssim \sup_{L=2^{-k}\leqslant 1} \|\zeta_L\|_{P_0+B(0,1)}^{2p} L^{2p(2-\alpha)}. \tag{2.6.8}$$

Proof. By splitting the interval $(0,1)$ into $[2^{-n}, 2^{-n+1})$ for $n \geqslant 1$, it is enough to prove that uniformly in $n \geqslant 1$ and $\lambda \in (0,1)$

$$\|\zeta_{2^{-n}(1+\lambda)}\| \lesssim 2^{-n(\alpha-2)} \sup_{L=2^{-k}\leqslant 1} L^{2-\alpha} \|\zeta_L\|. \tag{2.6.9}$$

This is a direct consequence of Lemma 2.8.1 quantifying the equivalence of Hölder norms coming from different kernels applied with $\Upsilon = \Psi_{2^{-n}(1+\lambda)}$ and $\theta = 2^{-m}$ for some integer m large enough. □

Proof of Lemma 2.6.1. This lemma is a variant of [55, Lemma 9] and we refer the reader to that lemma for the construction of a suitable modification of ζ. Here we only show the exponential integrability bound (2.6.7), using a similar argument as in [62, Lemma 4.1]. Throughout this proof, $\lesssim$ denotes a bound up to a constant that depends only on the dimension.

In the expansion in series of the exponential, we can exchange expectation and sum:

$$\mathbb{E}\Big[\exp\Big(\epsilon^2 \sup_{0<L\leqslant 1} \|(\zeta_{\frac{L}{2}})_{\frac{L}{2}}\|_{P_0}^2 L^{2(2-\alpha)} \Big) \Big] = \sum_{p=0}^{\infty} \epsilon^{2p} \frac{\mathbb{E}\left[\sup_{0\leqslant L\leqslant 1} \|(\zeta_{\frac{L}{2}})_{\frac{L}{2}}\|_{P_0}^{2p} L^{2p(2-\alpha)} \right]}{p!}.$$

Applying Lemma 2.6.4, we can bound the supremum over all L by the sum over dyadic L.

$$\mathbb{E}\left[\sup_{0<L\leqslant 1} \|(\zeta_{\frac{L}{2}})_{\frac{L}{2}}\|_{P_0}^{2p} L^{2p(2-\alpha)} \right] \leqslant \sum_{L=2^{-k}\leqslant \frac{1}{2}} \mathbb{E}\left[\|(\zeta_L)_L\|_{P_0+B(0,1)}^{2p} \right] L^{2p(2-\alpha)}.$$

Young's inequality implies

$$\|(\zeta_L)_L\|_{P_0+B(0,1)} \leqslant \|\zeta_L\|_{L^q, P_0+B(0,2)} \|\Psi_L\|_{L^{q'}},$$

where the subscript means that the L^q norm of ζ_L is taken over $P_0 + B(0,2)$ and where $q' = \frac{q-1}{q}$. By scaling, $\|\Psi_L\|_{L^{q'}} \lesssim L^{-\frac{d+2}{q}}$. We apply this with $q = 2p$.

$$\mathbb{E}\left[\|(\zeta_L)_L\|^{2p} \right] \lesssim \mathbb{E}\left[\|\zeta_L\|_{L^{2p}, P_0+B(0,2)}^{2p} \right] L^{-(d+2)}$$

$$= \mathbb{E}\left[\int_{P_0+B(0,2)} \zeta_L(t,x)^{2p} dtdx\right] L^{-(d+2)}$$
$$\lesssim \sup_{z\in P_0+B(0,2)} \mathbb{E}\left[\zeta_L(z)^{2p}\right] L^{-(d+2)}.$$

We bound $\mathbb{E}\left[\zeta_L(z)^{2p}\right]$ using the boundedness of σ. Without loss of generality, we show the computation for $z = (0,0)$. By the Burkholder-Davies-Gundy inequality,

$$\begin{aligned}
\mathbb{E}\left[\zeta_L(0,0)^{2p}\right] &= \mathbb{E}\left[\Big(\int_{(0,1)}\int_{\mathbb{R}^d} \Psi_L(t,x)\sigma(t,x)dW(t,x)\Big)^{2p}\right] \\
&\lesssim p^p\Big(\int\int\int \Psi_L(t,x)\Psi_L(t,x')\sigma(t,x)\sigma(t,x')K(x-x')dtdxdx'\Big)^p \\
&+ \mathbf{1}_{\{d=1,\lambda>1\}} p^p\Big(\int\int \Psi_L(t,x)^2\sigma(t,x)^2 dtdx\Big)^p \\
&\lesssim p^p(L^{-\lambda-2} + \mathbf{1}_{\{d=1,\lambda>1\}} L^{-d-2})^p \lesssim p^p L^{-p(\lambda+2)}.
\end{aligned}$$

We get that $\mathbb{E}\left[\|(\zeta)_L\|^{2p}\right] L^{2p(2-\alpha)} \lesssim p^p L^{p(2-2\alpha-\lambda)-(d+2)}$. Since $2-2\alpha-\lambda > 0$, for p large enough,

$$\sum_{L=2^{-k}\leqslant\frac{1}{2}} \mathbb{E}\left[\|(\zeta_L)_L\|^{2p}\right] L^{2p(2-\alpha)} \lesssim p^p \frac{1}{1-2^{-p(2-2\alpha-\lambda)+(d+2)}}.$$

By Stirling's formula, for p large enough,

$$\epsilon^{2p}\frac{\mathbb{E}\left[\sup_{0\leqslant L\leqslant 1}\|(\zeta_{\frac{L}{2}})_{\frac{L}{2}}\|^{2p} L^{2p(2-\alpha)}\right]}{p!} \lesssim \epsilon^{2p} e^p \sqrt{p},$$

hence for $\epsilon < e^{-2}$, (2.6.7) is verified. □

2.7 Invariant measure and Optimality

In this last section, we consider a special case of the SPDE considered in Section 2.6, namely the case of a one-dimensional reaction-diffusion equation driven by an additive space-time white noise. We aim to argue that in this case the bound obtained in Corollary 2.6.3 is optimal in terms of stochastic integrability.

Let $d = 1$ and let W be as in Section 2.6 with covariance operator $K\eta(x) = \eta(x)$. It is well-known [23, Section 11.2] that if we impose Dirichlet boundary conditions on the space-interval $[-1,1]$, then (2.6.5) defines a reversible Markov process with respect to the measure

$$\frac{1}{Z}\exp\left(-\int_{-1}^{1}\frac{1}{m+1}|u(x)|^{m+1}dx\right)\mu(du), \tag{2.7.1}$$

where μ is the law of an appropriately scaled Brownian bridge and Z is a normalisation constant. From the explicit expression (2.7.1) one can immediately read off that under this measure the following expectations are finite for $\alpha < \frac{1}{2}$ and ϵ small enough

$$\mathbb{E}\left[\exp\left(\epsilon\int_{-1}^{1}|u|^{m+1}dx\right)\right] < \infty \text{ and } \mathbb{E}\left[\exp\left(\epsilon[u]_\alpha^2\right)\right] < \infty. \tag{2.7.2}$$

The following proposition, the proof of which is given in Appendix 2.10, shows how to interpolate these two estimates to get optimal stochastic integrability for the supremum norm $\|u\|$.

Proposition 2.7.1. *If $u \in C^\alpha(-1,1)$ for $\alpha \in (0,1)$ and u^{m+1} is integrable, then u is bounded and we have the following interpolation:*

$$\Big(\frac{\|u\|_{(-1,1)}}{2}\Big)^{1+\alpha(m+1)} \leqslant \max\{[u]_\alpha\|u\|_{m+1}^{\alpha(m+1)}, \|u\|_{m+1}^{1+\alpha(m+1)}\}, \tag{2.7.3}$$

where $\|.\|_{m+1}$ refers to the L^{m+1} norm on $[-1,1]$.

Since $2\alpha < 1$,

$$\|u\|_{m+1}^{(m+1)\alpha}[u]_\alpha \leqslant \|u\|_{m+1}^{m+1} + [u]_\alpha^2.$$

Hence, (2.7.2) implies that for ϵ small,

$$\mathbb{E}\left[\exp\left(\epsilon\|u\|^{1+(m+1)\alpha}\right)\right] < \infty. \tag{2.7.4}$$

On the other hand, from Theorem 2.3.1 and from Corollary 2.6.3, we get

$$\mathbb{E}\Big[\exp\Big(\varepsilon(2^{-\frac{2}{m-1}}\|u\|_{P_{\frac{1}{2}}})^{2+(m-1)\alpha}\Big)\Big] < \infty. \tag{2.7.5}$$

Therefore, for $\alpha \to \frac{1}{2}$, the exponents in (2.7.4) and (2.7.5) both converge to $\frac{m+3}{2}$.

2.8 Appendix: Technical lemma

For a multi-index $m \in \mathbb{N}^{d+1}$, define $|m| = 2m_0 + \sum_{k=1}^d m_k$ the parabolic index, and $m! = \prod_{k=0}^d m_k!$.

Lemma 2.8.1. *Let Υ and Ψ be two C^∞ kernels supported in $B(0,R_\Upsilon)$ and $B(0,R_\Psi)$ respectively. Assume that for some odd integer $\beta > 0$, $\int_{\mathbb{R}^{d+1}} z^n\Psi(z) = \delta_{n=0}$ holds for all multi-indices n with $|n| \leqslant \beta$. For $R > 0$ set $\overline{R} := R + R_\Upsilon + 2R_\Psi$. Then for $\theta < \theta_0(\Psi)$ and for any function or distribution ζ*

$$\|\zeta * \Upsilon\|_{B(0,R)} \lesssim \sup_{|n|=\beta+1,\beta+2}\int|\partial^n\Upsilon|\sup_{L=\theta^k\leqslant 1}L^\beta\|\zeta * \Psi_L\|_{B(0,\overline{R})}. \tag{2.8.1}$$

Here and in the proof $\lesssim$ *means* $\leqslant C(\Psi, \beta)$.

Proof. Define inductively $\omega^0 = \Upsilon$ and $\omega^{k+1} = \omega^k - \Psi_{\theta^k} * \omega^k$. Since Ψ cancels all polynomials of degree less than β, one can see from Taylor's formula that

$$\int |\omega^k - \Psi_{\theta^k} * \omega^k| \lesssim \theta^{(\beta+1)k} \sup_{|n|=\beta+1,\beta+2} \int |\partial^n \omega^k|,$$

and

$$\int |\partial^n(\omega^k - \Psi_{\theta^k} * \omega^k)| \lesssim \int |\partial^n \omega^k|,$$

hence by induction

$$\int |\omega^k| \lesssim C^k \theta^{(\beta+1)k} \sup_{|n|=\beta+1,\beta+2} \int |\partial^n \Upsilon|. \tag{2.8.2}$$

This bound shows the convergence of the telescopic sum

$$\Upsilon = \sum_{k=0}^{\infty} \Psi_{\theta^k} * \omega^k \tag{2.8.3}$$

for θ small enough.

We can then write

$$\begin{aligned}
\|\zeta * \Upsilon\|_{B(0,R)} &= \Big\| \sum_{k=0}^{\infty} \Psi_{\theta^k} * \omega^k * \zeta \Big\|_{B(0,R)} \\
&\leqslant \sum_{k=0}^{\infty} \|\zeta * \Psi_{\theta^k}\|_{B(0,\overline{R})} \int |\omega^k| \\
&\leqslant \sup_{L=\theta^k \leqslant 1} L^\beta \|\zeta * \Psi_L\|_{B(0,\overline{R})} \sum_{k=0}^{\infty} C^k \theta^k \sup_{|n|=\beta+1,\beta+2} \int |\partial^n \Upsilon|.
\end{aligned}$$

□

The following lemma shows that the assumption of vanishing moments for Ψ can be removed from the previous lemma at the expense of making the domain on the right hand side even larger.

Lemma 2.8.2. *Let* Υ *and* Ψ *be two* C^∞ *kernels supported in* $B(0, R_\Upsilon)$ *and* $B(0, R_\Psi)$ *respectively. Let* $\beta > 0$ *be an odd integer and for* $R > 0$ *set* $\overline{R} := R + R_\Upsilon + 5R_\Psi$*.Then for* $\theta < \theta_0(\Psi)$ *and for any function or distribution* ζ

$$\|\zeta * \Upsilon\|_{B(0,R)} \lesssim \sup_{|n|=\beta+1,\beta+2} \int |\partial^n \Upsilon| \sup_{L=\theta^k \leqslant 1} L^\beta \|\zeta * \Psi_L\|_{B(0,\overline{R})}. \tag{2.8.4}$$

Here and in the proof $\lesssim$ *means* $\leqslant C(\Psi, \beta)$.

Proof. For any $\beta > 0$, we build from Ψ a kernel Ψ' that satisfies the hypothesis of Lemma 2.8.1.

We define $A_{n,m} = \int z^n \partial^m \Psi(z) dz$ and observe that since Ψ is compactly supported and since $\int \Psi = 1$, we have

$$A_{n,m} = \begin{cases} 0 & \text{if } |n| \leqslant |m|, n \neq m, \\ (-1)^{|m|} m! & \text{if } n = m. \end{cases}$$

Hence for any $\beta > 0$, $(A_{n,m})_{|n|,|m| \leqslant \beta}$ is an invertible linear system. By continuity of the coefficients, for r small enough,

$$A^r_{n,m} = \int z^n \partial^m (\Psi_r * \Psi)(z) dz \tag{2.8.5}$$

is also an invertible linear system. Hence, there exists coefficients $(a_m)_{|m| \leqslant \beta}$ such that

$$\sum_{|m| \leqslant \beta} a_m \int z^n \partial^m (\Psi_r * \Psi)(z) dz = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else.} \end{cases} \tag{2.8.6}$$

Set $\omega^{(0)} = \sum_{|m| \leqslant \beta} a_m \partial^m \Psi_r$ and $\Psi' = \omega^{(0)} * \Psi$, then

$$\int z^n \Psi'(z) dz = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{for } 0 < |n| < \beta. \end{cases} \tag{2.8.7}$$

We can therefore apply Lemma 2.8.1 with Ψ' to get

$$\|\zeta * \Upsilon\|_{B(0,R)} \lesssim \sup_{|n| = \beta+1, \beta+2} \int |\partial^n \Upsilon| \sup_{L = \theta^k \leqslant 1} L^\beta \|\zeta * \Psi_L\|_{B(0, R + R_\Upsilon + 2R_{\Psi'})}.$$

We have also

$$\begin{aligned} \|\zeta * \Psi'_L\|_{B(0, R+R_\Upsilon+2R_{\Psi'})} &= \|\zeta * \Psi_L * \omega^{(0)}_L\|_{B(0, R+R_\Upsilon+2R_{\Psi'})} \\ &\leqslant \|\zeta * \Psi_L\|_{B(0, R+R_\Upsilon+5R_\Psi)} \int |\omega^{(0)}|. \end{aligned}$$

□

2.9 Appendix: Low regularity Schauder estimate

We give here a proof of a low regularity Schauder estimate in our setting. This is enough for the current chapter. More elaborate Schauder estimmates are introduced in the following chapter.

Lemma 2.9.1. *Let u be a C^α-function for some $\alpha \in (0,1)$ and let $f := (\partial_t - \Delta)u$. There exists a constant $C = C(\alpha, d)$ such that*

$$[u]_\alpha \leqslant C \sup_{0<L<\infty} L^{2-\alpha}\|f_L\|. \tag{2.9.1}$$

Proof. Throughout the proof $\lesssim$ will denote a bound up to a multiplicative constant, which may change from line to line, but which always depends only on α and d. Define $N = \sup_{L\leqslant 1} L^{2-\alpha}\|f_L\|$. Since $(\cdot)_L$ denotes the convolution with a smooth kernel, it commutes with derivatives. We know that for $L < 1$, for any $l \in \text{span}\{1, x_i, i \in \{1, ..., d\}\}$, we have on $\mathbb{R} \times \mathbb{R}^d$,

$$(\partial_t - \Delta)(u_L - l) = f_L.$$

For $z_0 \in B(0,1)$, for some $\delta > 0$ to be fixed below, define $v_>$ as the solution to

$$(\partial_t - \Delta)v_> = \mathbf{1}_{\{B(z_0,\delta)\}} f_L, \quad v_>|_{\partial B(z_0,\delta)} = 0,$$

where $\partial B(z_0, \delta) = \{z = (t, x), d(z, z_0) = \delta, t \leqslant t_0\}$ is the parabolic boundary of $B(z_0, \delta)$. The first interesting inequality we get from standard heat equation estimates [49, Cor.8.1.5] is

$$\|v_>\| \lesssim \delta^2\|f_L\| \leqslant \delta^2 L^{\alpha-2} N. \tag{2.9.2}$$

Define $v_< = u_L - v_>$. As $(\partial_t - \Delta)v_< = 0$ on $B(z_0, \delta)$ for any differential operator $D \in \{\partial_t, \partial_i\partial_j, i, j \in \{1, ..., d\}\}$,

$$\|Dv_<\|_{B_{\frac{\delta}{2}}} \lesssim \delta^{-2} \inf_l \|u_L - l\|_{B(z_0,\delta)},$$

where l runs over all function spanned by 1 and $x_i, i \in \{1, ..., d\}$. Therefore, for any $R < \frac{\delta}{2}$, for the same range of operator D, for a suitably chosen $l_R \in \text{span}\{1, x_i, i \in \{1, ..., d\}\}$,

$$\|v_< - l_R\|_{B(z_0,R)} \leqslant R^2\|Dv_<\|_{B(z_0,R)} \lesssim \left(\frac{R}{\delta}\right)^2 \inf_l \|u_L - l\|_{B(z_0,\delta)}.$$

Using the definition of $v_<$ and the triangle inequality,

$$\|u_L - l_R\|_{B(z_0,R)} - \|v_>\|_{B(z_0,R)} \lesssim \left(\frac{R}{\delta}\right)^2 \inf_l \|u_L - l\|_{B(z_0,\delta)}.$$

From (2.9.2),

$$\frac{1}{R^\alpha}\|u_L - l_R\| \lesssim \left(\frac{R}{\delta}\right)^{2-\alpha} \frac{1}{\delta^\alpha} \inf_l \|u_L - l\|_{B(z_0,\delta)} + \left(\frac{\delta}{L}\right)^2 \left(\frac{L}{R}\right)^\alpha N. \tag{2.9.3}$$

Furthermore, from (2.3.6) we get,

$$\frac{1}{R^\alpha}\|u - l_R\|_{B(z_0,R)} \lesssim \frac{1}{R^\alpha}\|u_L - l_R\|_{B(z_0,R)} + \left(\frac{L}{R}\right)^\alpha [u]_\alpha. \tag{2.9.4}$$

Similarly, for any $l \in \text{span}\{1, x_i, i \in \{1, ..., d\}\}$

$$\frac{1}{\delta^\alpha}\|u_L - l\|_{B(z_0,\delta)} \lesssim \frac{1}{\delta^\alpha}\|u - l\|_{B(z_0,\delta)} + \left(\frac{L}{\delta}\right)^\alpha [u]_\alpha. \tag{2.9.5}$$

Hence for $0 < \epsilon < 1$, for $L = \epsilon R = \epsilon^2 \delta$, (2.9.3) and (2.9.4),(2.9.5) give:

$$\frac{1}{R^\alpha} \inf_l \|u - l\|_{B(z_0,R)} \lesssim \epsilon^{2-\alpha} \frac{1}{\delta^\alpha} \inf_l \|u - l\|_{B(z_0,\delta)} \tag{2.9.6}$$

$$+(\epsilon^\alpha + \epsilon^{2\alpha})[u]_\alpha + \epsilon^{\alpha-4} N. \tag{2.9.7}$$

Note that $[u]_\alpha \sim \sup_{z_0} \sup_\delta \frac{1}{\delta^\alpha} \inf_l \|u - l\|_{B(z_0,\delta)}$, hence

$$[u]_\alpha \lesssim (\epsilon^{2-\alpha} + \epsilon^\alpha + \epsilon^{2\alpha})[u]_\alpha + \epsilon^{\alpha-4} N. \tag{2.9.8}$$

By making ϵ small enough, we can absorb $[u]_\alpha$ in the right-hand side of (2.9.8) into the left-hand side, concluding the proof of the Schauder estimate (2.9.1).

□

Corollary 2.9.2. *Let f be compactly supported in $B(0,R)$ and let u be the unique bounded solution to $(\partial_t - \Delta)u = f$ which vanishes for $t \leqslant -R^2$. Then for $\alpha \in (0,1)$ there exists a constant $C = C(\alpha, d)$ such that*

$$[u]_\alpha \leqslant C \sup_{0<L\leqslant 2R} L^{2-\alpha} \|f_L\|, \tag{2.9.9}$$

and

$$\|u\| \leqslant CR^{\alpha} \sup_{L \leqslant 2R} L^{2-\alpha} \|f_L\|. \tag{2.9.10}$$

Proof. By a scaling argument, we can show the result for $R = \frac{1}{2}$. We first show an equivalence of kernels by applying Lemma 2.8.2 with $\beta > 1-\alpha$ and $\Upsilon = \Psi_L - \Psi_* \Psi_{L-1}$. Since f has compact support, we do not need to keep track of the domains. There exists a $\theta < 1$ and a constant $C = C(\Psi, \beta)$ such that:

$$\|f_L - (f_1)_{L-1}\| \leqslant C \sup_{|n|=\beta+1,\beta+2} \int |\partial^n (\Psi_L - \Psi * \Psi_{L-1})| \sup_{S=\theta^k} S^{\beta} \|f_S\|$$

Therefore, we can write

$$\begin{aligned} L^{2-\alpha}\|f\|_L &\leqslant L^{2-\alpha}\|(f_1)_{L-1}\| + L^{2-\alpha} C L^{-\beta-1} \sup_{S=\theta^k} S^{\beta}\|f_S\| \\ &\leqslant L^{2-\alpha}(L-1)^{-d-2}\|f_1\| + CL^{1-\alpha-\beta} \sup_{S \leqslant 1} S^{2-\alpha}\|f_S\| \\ &\leqslant \left(L^{2-\alpha}(L-1)^{-d-2} + CL^{1-\alpha-\beta}\right) \sup_{S \leqslant 1} S^{2-\alpha}\|f_S\|. \end{aligned}$$

Therefore the supremum can be taken over scales $L \leqslant 2R$. In the proof of Lemma 2.9.1, the only place where the hypothesis $[u]_{\alpha} < \infty$ was used was in (2.9.8).This assumption can be removed as by regularising the equation first, we have that uniformly for any $\tau > 0$,

$$[u_\tau]_{\alpha} \lesssim \sup_{L \leqslant 2R} L^{2-\alpha} \|(f_\tau)_L\| \leqslant \sup_{L \leqslant 2R} L^{2-\alpha} \|f_L\|,$$

and as u is continuous, we can pass to the limit for $\tau \to 0$.

For (2.9.10), we write $u = u_R + (u - u_R)$. For the second term we get

$$\|u - u_R\| \leqslant R^{\alpha}[u]_{\alpha} \overset{(2.9.9)}{\lesssim} R^{\alpha} \sup_{0<L\leqslant 2R} L^{2-\alpha}\|f_L\|.$$

For the smooth part u_R we write $(\partial_t - \Delta)u_R = f_R$ we use a standard L^{∞} estimate for the heat equation with compactly supported right-hand side [49, Thm 8.4.2] and get

$$\|u_R\| \lesssim R^2 \|f_R\| \lesssim R^{\alpha} \sup_{0<L\leqslant 2R} L^{2-\alpha}\|f_L\|.$$

□

2.10 Appendix: Proof of Proposition 2.7.1

If $u \in C^\alpha(-1,1)$ for $\alpha \in (0,1)$ and u^{m+1} is integrable, then u is bounded and we have the following interpolation:

$$\Big(\frac{\|u\|_{(-1,1)}}{2}\Big)^{1+\alpha(m+1)} \leqslant \max\{[u]_\alpha \|u\|_{m+1}^{\alpha(m+1)}, \|u\|_{m+1}^{1+\alpha(m+1)}\}, \tag{2.10.1}$$

where $\|.\|_{m+1}$ refers to the L^{m+1} norm on $[-1,1]$.

For any interval $I \subset [-1,1]$,

$$|u(t)| - \frac{1}{|I|}\Big|\int_I u(s)ds\Big| \leqslant \frac{1}{|I|}\int_I |u(t)-u(s)|ds \leqslant \frac{1}{|I|}\int_I [u]_{\alpha,I}|t-s|^\alpha.$$

If $t \in I$, $|t-s|^\alpha \leqslant I^\alpha$. We can apply Jensen's inequality.

$$|u(t)| \leqslant \left(\frac{1}{|I|}\int_I |u(s)|^{m+1}ds\right)^{\frac{1}{m+1}} + |I|^\alpha [u]_{\alpha,I}.$$

And since this is true for any $I \subset [-1,1]$, we have for any choice of $0 < x \leqslant 2$,

$$\|u\|_I \leqslant x^{-\frac{1}{m+1}}\|u\|_{m+1} + x^\alpha [u]_{\alpha,I}.$$

If $\|u\|_{m+1} \geqslant [u]_\alpha$ then choose $x = 1$ to get $\|u\|_{(-1,1)} \leqslant 2\|u\|_{m+1}$. Else choose $x = (\|u\|_{m+1}/[u]_\alpha)^{\frac{m+1}{1+\alpha(m+1)}} \leqslant 1$ and get $\|u\|_{(-1,1)} \leqslant 2[u]_\alpha^{\frac{1}{1+\alpha(m+1)}}\|u\|_{m+1}^{\frac{\alpha(m+1)}{1+\alpha(m+1)}}$. In conclusion,

$$\|u\|_{(-1,1)} \leqslant 2\max\{\|u\|_{m+1}, [u]_\alpha^{\frac{1}{1+\alpha(m+1)}}\|u\|_{m+1}^{\frac{\alpha(m+1)}{1+\alpha(m+1)}}\}.$$

Chapter 3

Space-time localisation for Φ^4_3

3.1 Introduction

The aim of this chapter is to understand how techniques from the previous chapter can be adapted to derive a priori bounds for the three dimensional stochastic quantisation equation, also known as the dynamic Φ^4_3 model. This model is - at least formally - given by the nonlinear stochastic partial differential equation (SPDE)

$$(\partial_t - \Delta)u = -u^3 + \xi, \tag{3.1.1}$$

where ξ is the space-time white noise over $\mathbb{R} \times \mathbb{R}^d$. In our main result, Theorem 3.2.1, we show a bound on the solution in the case $d = 3$ on a compact space-time set that depends only on a finite number of explicit polynomials in the Gaussian noise on a slightly larger space-time set. In particular, our bound does not depend on any space-time boundary conditions.

The main difficulty when working with (3.1.1) compared to the previous chapter is the roughness of the driving noise ξ which in turn makes the solution irregular and the interpretation of nonlinear terms non-trivial. It is now well-understood that solutions are distribution valued in spatial dimension $d \geqslant 2$ and the nonlinearity has to be renormalised, which loosely speaking corresponds to replacing (3.1.1) by

$$(\partial_t - \Delta)u = -u^3 + \text{“}\infty\text{”}\ u + \xi. \tag{3.1.2}$$

The theory of singular SPDEs of this type has been revolutionised in the recent years, starting with Hairer's theory of regularity structures [41].

This theory is by now well-developed and enables the analysis of a range of equations which are much more singular than the dynamic Φ^4_3 model, however we recall that the

arguments currently available are insufficient to go beyond a short time existence theory in any of these equations. For example, the construction of solutions to (3.1.1) in [41] does not make use of the "good" sign of the nonlinear term $-u^3$ and would work equally if it were replaced with a $+u^3$. Solutions for this modified equation are expected to blow up in finite time.

For (3.1.1) in dimension 3 the problem of passing from a local to a global solution theory has been largely overcome in a series of very recent works starting with [54] where (3.1.1) was studied on the torus $\mathbb{T}^3$ and a priori estimates were obtained which ruled out the possibility of finite time blow-up. In [35] a priori estimates for solutions on the full space $\mathbb{R}^3$ were shown; see also [2, 36] for an analysis of the invariant measures based on similar ideas. All of these articles worked in the framework of *paracontrolled distribution* rather than regularity structures.

In this chapter we adapt the technique combining the maximum principle with the convolution argument from the previous chapter to derive a priori estimates within the framework of regularity structures. We show a space-time version of the "coming down from infinity" property, i.e. we provide a bound on the solution on a compact space-time set that depends on the realisation of the noise on a slightly larger set, but does not depend on the behaviour of the solution elsewhere, making full use of the strong nonlinear damping term $-u^3$. This local dependence makes this bound extremely useful when analysing the behaviour of solutions on large scales.

A main interest of this approach is the technique itself. Its advantages are that we effectively separate the argument for small and large scales by dealing with a family of regularised equations for large scales and use (an appropriate restatement) of the theory of regularity structures to analyse the small scales. This results in a relatively short argument compared to previous works and has the potential to work for a much larger class of equations. We want to stress that our argument does not make use explicitly of any of the results in [41]. In fact, in both the statement of our main result and its proof we fully avoid the terminology of this theory, i.e. the notions of model, modelled distribution, structure group etc. but give a direct statement of all of the required bounds. This is possible, because the algebra involved in the small scale solution theory of (3.1.1) is still not too complex and we hope that our direct approach makes the presentation more clear. We do however include a separate section in which we translate our main estimates into the regularity structure terminology. This may also be used as a practical example to understand the more complete introduction of regularity structures in the next chapter.

This chapter is structured as follows. Section 3.2 contains the elements needed to state our main result, Theorem 3.2.1, starting with the definitions of the proper Hölder spaces

in Section 3.2.1. Section 3.2.2 presents the setting in which we solve Equation (3.1.1) and our main result. The outline of the proof and the different lemmas required are presented in Section 3.2.3. We then explain the close connection between our setting and Hairer's theory of regularity structures in Section 3.3, where the full regularity structure for the Φ^4_3 equation is presented. Section 3.4 contains the proof of the Theorem 3.2.1.

3.2 Setting and main result

3.2.1 Measuring regularity

As in chapter 2, regularity will be measured with respect to the metric

$$d((t,x),(\overline{t},\overline{x})) = \max\Big\{\sqrt{|t-\overline{t}|}, |x-\overline{x}|\Big\}, \tag{3.2.1}$$

and we use the same balls introduced previously (2.3.2).

Since we are dealing with a wider range of regularities, we need to expand the definitions of Hölder space. For $\alpha \in (0,1)$, we keep the same definition of the Hölder semi-norm $[.]_\alpha$

$$[u]_\alpha := \sup_{z\neq\overline{z}\in\mathbb{R}\times\mathbb{R}^3} \frac{|u(z)-u(\overline{z})|}{d(z,\overline{z})^\alpha}. \tag{3.2.2}$$

For $\alpha \in (1,2)$, we define the Hölder semi-norm $[.]_\alpha$

$$[u]_\alpha := \sup_{z\neq\overline{z}\in\mathbb{R}\times\mathbb{R}^3} \frac{|u(z)-u(\overline{z})-\nabla u(z).X(z-\overline{z})|}{d(z,\overline{z})^\alpha}, \tag{3.2.3}$$

where ∇ refers to the spatial gradient, and we introduce the function X which is the projection on space coordinates. We will often deal with functions $U(z,\overline{z})$ of two variables generalising the increments of $u(z)-u(\overline{z})$ in (3.2.3) above. In this case we define for $\alpha \in (1,2)$

$$[U]_\alpha := \sup_{z\in\mathbb{R}\times\mathbb{R}^3} \inf_{\nu(z)\in\mathbb{R}^3} \sup_{\overline{z}\in\mathbb{R}\times\mathbb{R}^3\setminus\{z\}} \frac{|U(\overline{z},z)-\nu(z).X(z-\overline{z})|}{d(z,\overline{z})^\alpha}. \tag{3.2.4}$$

The infimum over functions ν is attained when $\nu(z)$ is the spatial gradient in the second coordinate of U at point (z,z). As before, we often work with norms which only depend on the behaviour of functions / distributions on a fixed subset of time-space: if $B \subset \mathbb{R}\times\mathbb{R}^3$ is a bounded set, then we define the local α-Hölder semi-norm $[.]_{\alpha,B}$ as in (3.2.2) with the supremum restricted to $z,\overline{z} \in B$. The use of a third index r as in $[.]_{\alpha,B,r}$ indicates that the supremum is restricted to z and $\overline{z}$ at distance at most r. Similarly, $\|.\|$ denotes the L^∞ norm on the whole space $\mathbb{R}\times\mathbb{R}^3$ and $\|.\|_B$ the norm of the restriction of the function to B, and for a function of two variable, $\|.\|_{B,r}$ is the norm restricted to

$z, \bar{z} \in B$ with $d(z, \bar{z}) \leqslant r$.

From now on and for the remainder of this chapter, x, y and z will always denote a generic space-time variable.

In the spirit of the convolution method, we work again with a Besov-Hölder type norm to measure negative regularity, however we need here further precision in the choice of the kernel. Our definition is strongly inspired by the choice of smooth kernel satisfying the semi-group property with respect to the scaling parameter first introduced in [62]. This semi-group property allows us to effectively connect regularisations at different scales and thus makes the proof of the Reconstruction Theorem 3.2.8 very convenient. However, an additional twist is required. For us it is important to be able to define local norms that only depend on properties of distributions on a compact set. This makes it most convenient to work with a compactly supported kernel in the definition of the norm. But the kernel used in [62] does not have this property. Wavelet bases, on the other hand, permit a convenient transition from one scale to another and can consist of compactly supported functions, but unfortunately the projection on these basis functions do not commute with differential operators. The following simple construction yields a kernel which is compactly supported and enjoys a version of the semi-group property for dyadic scales which is enough to prove the reconstruction theorem.

We fix a non-negative smooth function Φ with support in $B(0,1)$, symmetric in space, with $\Phi(x) \in [0,1]$ for all $x \in \mathbb{R} \times \mathbb{R}^3$ and with integral 1. Setting $\Phi_L(t,x) = L^{-5}\Phi(\frac{t}{L^2}, \frac{x}{L})$, we now define $\Psi_{L,n} = \Phi_{L2^{-1}} * \Phi_{L2^{-2}} * \ldots * \Phi_{L2^{-n}}$ and $\Psi_L = \lim_{n\to\infty} \Psi_{L,n}$ so that $\Psi_L = \Phi_{\frac{L}{2}} * \Psi_{\frac{L}{2}}$. The convergence can be checked easily. Ψ_L and $\Psi_{L,n}$ are non-negative and smooth, symmetric in space and with support $B(0,1)$ and $B(0, 1-2^{-n})$. We define the operator $(\cdot)_L$ by convolution with Ψ_L, and $(\cdot)_{L,n}$ by convolution with $\Psi_{L,n}$ for $n \geqslant 1$. $(\cdot)_{L,0}$ is the identity. Since $\Psi_{L,n+m} = \Psi_{L,n} * \Psi_{L2^{-n},m}$, we have

$$(\cdot)_{L,n+m} = ((\cdot)_{L2^{-n},m})_{L,n}. \tag{3.2.5}$$

Taking m to infinity in this, or equivalently noticing that $\Psi_L = \Psi_{L,n} * \Psi_{L2^{-n}}$, we have the desired relation between dyadic scales

$$(\cdot)_L = ((\cdot)_{L2^{-n}})_{L,n}. \tag{3.2.6}$$

We then define the local C^α norm of a distribution θ for $\alpha < 0$ as

$$[\theta]_{\alpha,C} = \sup_{L\leqslant 1} \|(\theta)_L\|_C L^{-\alpha}. \tag{3.2.7}$$

It is proven in [5, Theorem 2.34] that for a similar quantity, in the case where C is a

torus of size one this corresponds to the classical Besov norm $\mathcal{B}^{\alpha}_{\infty,\infty}$. In our case, $[\theta]_{\alpha,C}$ depends on the distribution θ on $C + B(0,1)$ since Ψ has support in $B(0,1)$

Furthermore, we recall and mention the scaling estimates, for $n \in \mathbb{N} \cup \{\infty\}$ and $\alpha > -5$

$$\int |\Psi_{L,n}(x-y)| d(x,y)^{\alpha} dy \leqslant L^{\alpha}, \quad \int |\nabla \Psi_{L,n}(x-y)| d(x,y)^{\alpha} dy \lesssim L^{\alpha-1}. \tag{3.2.8}$$

Here and in the rest of the thesis, "$\lesssim$" denotes a bound that holds up to a multiplicative constant. This immediately implies that for any $h \in C^{\alpha}$, $\alpha \in (0,2)$, and for any bounded set C, we have

$$\|h_L - h\|_{C_L} \leqslant L^{\alpha} \sup_{z \in C_L} [h]_{\alpha, B(z,L)} = L^{\alpha} [h]_{\alpha, C, 2L}. \tag{3.2.9}$$

Indeed, since Ψ is symmetric in space we have $\int \Psi(y) X(y) dy = 0$ where X denotes the projection onto space coordinates, and for all $x \in C$:

$$\begin{aligned}
(h_L - h)(x) &= \int \Psi_L(x-y)(h(y) - h(x)) dy \\
&= \int \Psi_L(x-y)(h(y) - h(x) - \mathbf{1}_{\{\alpha>1\}} \nabla h(x).X(y-x)) dy \\
&\lesssim [h]_{\alpha, B(x,L)} \int \Psi_L(x-y) d(x,y)^{\alpha} dy.
\end{aligned}$$

For products of functions, we will sometimes be using the following notational convention:

$$(fg)_L(x) - f(x) g_L(x) = ((f - f(x))g)_L(x).$$

The presence of the variable means that we evaluate the function there first, and the absence means that the convolution variable is used.

3.2.2 Main result

We will work with a regularised version of (3.1.1) throughout, i.e we assume that u is a smooth function which on P satisfies

$$(\partial_t - \Delta) u = -u^3 + \zeta + (3C_1 - 9C_2) u, \tag{3.2.10}$$

for real valued parameters C_1, C_2. Thus, throughout this work, we never have to address the question of how a given expression has to be interpreted to make sense. The main application we have in mind the case where $\zeta = \xi_{\delta}$, i.e. a regularisation of the white noise at scale δ and where C_1 and C_2 are defined as the expectations of certain polynomials in ξ_{δ} which diverge like $\frac{1}{\delta}$ and $\log \delta^{-1}$ as the regularisation is removed. More precision on

the meaning of these constants was provided in section 1.2.3. However, in our analysis these values only enter in the assumptions on the "trees" (see (3.2.12), (3.2.15) and (3.2.17)) and their precise values do not appear. Despite dealing with smooth functions we stress that all of our estimates are stable in the limit $\delta \to 0$, where ξ can only be measured as a distribution of regularity $-\frac{5}{2}-$ and u as a distribution of regularity $-\frac{1}{2}-$. We will freely use the convention to speak of "distributions" when we refer to smooth functions that can only be measured in a distributional norm in this limit.

We first introduce several polynomials in ζ that are used in the local description of the solution to (3.2.10). These are (essentially) the same objects which appear in Hairer's small scale solution theory for (3.2.10) and we use his convention to denote these objects by trees.

We start by fixing an $\varepsilon > 0$ which will always be assumed to be "sufficiently small". The first tree is $\Irightarrow$ is assumed to satisfy the point-wise identity on P

$$(\partial_t - \Delta)\Irightarrow = \zeta, \tag{3.2.11}$$

and we assume a control in the $C^{-\frac{1}{2}-\epsilon}$ norm. The constant C_1 appears in the following definitions of the trees $\Vee$ and $\Vvee$:

$$\Vee := \Irightarrow^2 - C_1, \quad \Vvee := \Irightarrow^3 - 3C_1\Irightarrow, \tag{3.2.12}$$

and these distributions will be measured in the norms of $C^{-1-2\varepsilon}$ and $C^{-\frac{3}{2}-3\varepsilon}$, respectively. We also introduce symbols of higher order: we assume that $\Yup$ and $\Yvee$ satisfy the point-wise identity on P

$$(\partial_t - \Delta)\Yup = \Vee, \quad (\partial_t - \Delta)\Yvee = \Vvee. \tag{3.2.13}$$

As expected with the heat operator, we assume that the regularity is increased by 2 i.e. $\Yup \in C^{1-2\epsilon}$ and $\Yvee \in C^{\frac{1}{2}-3\epsilon}$. The trees are built with leaves $\bullet$ which represents the noise ξ, solving the heat equation is represented by a vertical edge, and taking renormalised products of trees is done by joining them at the root.

Finally we introduce the trees $\Vee_x$ denoting the product of $\Vee$ with X, $\Vee\Yup$, $\Vee\Yvee$ and $\Vee\Yup^2$ and for these we will need bounds on the quantities

$$[\Vee_x]_{-2\epsilon} = \sup_{x\in P}\sup_{L<1} L^{2\epsilon}\Big|\int X(y-x)\Vee(y)\Psi_L(y-x)dy\Big|, \tag{3.2.14}$$

$$[\Vee\Yup]_{-4\epsilon} = \sup_{x\in P}\sup_{L<1} L^{4\epsilon}\Big|\int ((\Yup(y)-\Yup(x))\Vee(y) - C_2)\Psi_L(y-x)dy\Big|, \tag{3.2.15}$$

$$[\Vee\Yup^2]_{-4\epsilon} = \sup_{x\in P}\sup_{L<1} L^{4\epsilon}\Big|\int (\Yup(y)\Irightarrow(y) - \Yup(x)\Irightarrow(y))\Psi_L(y-x)dy\Big|, \tag{3.2.16}$$

$$[\varXi]_{-\frac{1}{2}-5\epsilon} = \sup_{x\in P}\sup_{L<1} L^{\frac{1}{2}+5\epsilon}\Big|\int ((\varUpsilon(y)-\varUpsilon(x))\varPsi(y) - 3C_2\varXi(y))\Psi_L(y-x)dy\Big|. \tag{3.2.17}$$

We will work with the function $v := u - \varXi$ which satisfies

$$\begin{aligned}(\partial_t - \Delta)v =& -u^3 + (3C_1 - 9C_2)u = -(v+\varXi)^3 + (3C_1-9C_2)(v+\varXi)\\ =& -v^3 - 3v^2\varXi - 3v(\varXi^2 - C_1) - (\varXi^3 - 3C_1\varXi) - 9C_2(v+\varXi)\\ =& -v^3 - 3v^2\varXi - 3v\varV - \varW - 9C_2(v+\varXi).\end{aligned} \tag{3.2.18}$$

The fact that the constant C_1 disappears in this expansion was already noted in [22] and that was enough to define solutions in dimension 2, where the constant C_2 is unnecessary.

The main result can now be stated.

Theorem 3.2.1. *If v solves* (3.2.18) *pointwise on P, then we have:*

$$\|v\|_{P_R} \leqslant C\max\Big\{\frac{1}{R}, [\tau]_{|\tau|}^{\frac{1}{n_\tau(\frac{1}{2}-\epsilon)}}, \tau\in\mathcal{T}\Big\}, \tag{3.2.19}$$

where $\mathcal{T} = \{\varXi, \varV, \varV_{\circ}, \varUpsilon, \varW, \varUpsilon, \varXi, \varXi, \varXi\}$, $|\tau|$ is the regularity in which we measure the tree τ in the way explained above and n_τ is the number of noises or leaves appearing in the tree.

Remark 3.2.2. As stated above, the bounds we assume on the "trees" are (almost – see the following Remarks 3.2.4 and 3.2.5) identical to those appearing as input into the analytic part of [41]. The particular form of the x-dependent "counterterms" $-X(x)\varV(y)$ in (3.2.14), $\varUpsilon(x)\varV(y)$ in (3.2.15), $-\varUpsilon(x)\varV(y)$ in (3.2.16) and $\varUpsilon(x)\varV(y)$ in (3.2.17) corresponds exactly to the "positive renormalisation" or re-centring procedure of the trees performed there. See Section 3.3 for a more detailed discussion of positive renormalisation in the theory of regularity structures.

In the case where $\zeta = \xi_\delta$ is a regularised white noise and where $C_1 = \mathbb{E}\varXi(y)^2$ and $C_2 = \mathbb{E}\varUpsilon(y)\varV(y)$, e.g. for $y = (1,0)$, uniform-in-δ-bounds on the various norms were obtained in [41, Section 10]. We stress that in this low-regularity situation the convergence of these terms as $\delta\to 0$ is highly non-obvious, even after renormalisation. The calculations use probabilistic tools and strongly rely on stochastic cancellations.

The estimates in [41, Section 10] actually yield bounds on the moments of all of these terms, so that our main result (3.2.19) implies bounds on moments of the solution: Since τ is a random variable in the (inhomogeneous) Wiener chaos of order n_τ over the

Gaussian noise, $[\tau]_{\gamma_\tau}$ has Gaussian tails, more precisely [27]:

$$\mathbb{E}[\exp(\lambda[\tau]_{\gamma_\tau}^{\frac{2}{n_\tau}})] < \infty,$$

for some $\lambda > 0$. Hence for $\overline{\lambda} = \frac{\lambda}{C^{1-2\epsilon}}$ we get

$$\mathbb{E}[\exp(\overline{\lambda}\|v\|_{P_R}^{1-2\epsilon})] < \infty. \tag{3.2.20}$$

Remark 3.2.3. One of the main motivations to consider (3.1.1) is to use the Markovian dynamics described by it to study its invariant measure, the Euclidean Φ_3^4 quantum field theory. In order to link this Euclidean (imaginary time) field theory to a real time field theory, this measure should satisfy certain properties, the Osterwalder-Schrader axioms [32, section 6.1]. Our bound (3.2.20) immediately transfers to this invariant measure. Unfortunately, these stretched exponential moments just fall short of the exponential bounds required for the Analyticity Axiom.

Remark 3.2.4. Hairer's convention in the definition of the symbols $\text{\textexclamdown}$ in (3.2.11), Υ and $\Upsilon^{\bullet}$ in (3.2.13) differs slightly from ours: Instead of assuming that these objects satisfy a partial differential equation as we do, he defines them using an integral condition, e.g.

$$\text{\textexclamdown}(x) = \int_{\mathbb{R}\times\mathbb{R}^3} K(y,x)\zeta(y)dy,$$

for a singular integral kernel K. This kernel K is essentially the Gaussian heat kernel, but it is post-processed to make it compactly supported and to integrate to 0 against polynomials up to a certain degree. After this post-processing K is not associated to a differential operator any more and in this definition $\text{\textexclamdown}$ and the other stochastic terms are not characterised by a (simple) PDE. This is in line with the general philosophy pursued in [41] to view (3.1.1) as an integral equation using the mild formulation rather than a differential equation.

Remark 3.2.5. Continuing the discussion of the symbols $\text{\textexclamdown}$, Υ and $\Upsilon^{\bullet}$ we point out that in (3.2.11) and (3.2.13) we do not impose boundary conditions, but only that a certain PDE holds point-wise. There is thus some choice in how these objects are defined and our main result, the estimate (3.2.19), holds uniformly over all of these choices. This is also the reason why the symbols Υ and $\Upsilon^{\bullet}$ appear in the list $\mathcal{T}$. For many choices of boundary conditions Schauder theory would imply $[\Upsilon]_{1-2\epsilon} \lesssim [\Upsilon]_{-1-2\epsilon}$ and $[\Upsilon^{\bullet}]_{\frac{1}{2}-3\epsilon} \lesssim [\Upsilon^{\bullet}]_{-\frac{3}{2}-3\epsilon}$ so that these symbols could be removed from $\mathcal{T}$.

A natural choice to would be to impose Dirichlet boundary conditions on the parabolic boundary of P in (3.2.11) and (3.2.13) and in this case such a Schauder estimate holds indeed. Moreover, with this choice one would have the nice property that all of the

objects on the right-hand side only depend on the realisation of ζ on P, which would be in line with a "space-time Markov property." This nice choice has the slight disadvantage that (in the case where $\zeta = \xi_\delta$ is a regularised white noise) the negative renormalisation would have to be modified reflecting the boundary conditions which would lead to x-dependent C_1 and C_2 in (3.2.12), (3.2.15), (3.2.17), and then an extra term would have to be added in (3.2.10) in order to make the renormalisation of the original equation x-independent. Such a construction could certainly be implemented, but we refrain from doing so here (see [30], [68] for discussion of similar boundary issues).

Remark 3.2.6. The spatial dimension $d = 3$ only enters our analysis through the regularity assumptions on the "trees". The various $|\tau|$ are all derived from the parabolic regularity of the white noise in $1 + 3$ time-space dimensions , which is $-\frac{5}{2}-$. The actual PDE arguments we present do not rely on a specific choice of d.

3.2.3 Outline of proof

One of the key ideas behind the theory of regularity structures is the following scaling argument:

$$\widehat{u}(t, x) = \lambda^{\frac{d}{2}-1} u(\lambda^2 t, \lambda x) \tag{3.2.21}$$

is the scaling under which the stochastic heat equation $(\partial_t - \Delta)u = \xi$ is invariant in law. For the Φ^4 equation the nonlinearity $-u^3$ scales like $-\lambda^{4-d}\widehat{u}^3$. In dimension less than 4, this term formally vanishes on small scales, i.e. when λ goes to zero. This property is called subcriticality in Hairer's theory and corresponds to super-renormalisability in quantum field theory. This observation suggests that in order to control the behaviour of u on "small scales" one should use the heat operator and treat the nonlinearity as a perturbation. This is precisely how a small-scale local solution theory is built in [41]. The sign of the nonlinearity $-u^3$ is not used in this argument. The argument for large scales on the other hand clearly has to rely on the "good term" $-u^3$ and should not use the smoothing of the heat operator too much.

We have already seen that as a perturbation of the linear equation, $v = u - \varolessthan$ satisfies

$$(\partial_t - \Delta)v = -v^3 - 3v^2\varolessthan - 3v\varwedge - \varvee - 9C_2(v + \varolessthan). \tag{3.2.22}$$

To control large scales, we apply the regularising operator $(\cdot)_L$ for some L to be chosen below and we get the equation

$$(\partial_t - \Delta)v_L = -(v_L)^3 - 3(v^2\varolessthan)_L - 3(v\varwedge)_L - 9C_2(v_L + \varolessthan_L) - (\varvee)_L + ((v_L)^3 - (v^3)_L). \tag{3.2.23}$$

This equation is not closed in terms of v_L and we will require control on the commutator

$(v_L)^3 - (v^3)_L$ and on the products $(v^2\varXi)_L$ and $(v\mathbb{V})_L$. These are bounded in the small-scale theory. For large scale bounds, we use the following lemma

Lemma 3.2.7. *Let u be a continuous function defined on $[0,1] \times [-1,1]^3$, for which the following holds point-wise in $(0,1] \times (-1,1)^3$:*

$$(\partial_t - \Delta)u = -u^3 + g(u, z), \tag{3.2.24}$$

where g is a bounded function. We have the following point-wise bound on u, for all $(t,x) \in (0,1] \times (-1,1)^3$:

$$|u(x,t)| \leqslant C \max\left\{ \frac{1}{\min\{\sqrt{t}, (1-x_i), (1+x_i), i = 1,2,3\}}, \|g\|^{\frac{1}{3}} \right\}, \tag{3.2.25}$$

for some independent constant C.

This lemma is a simplified version of 2.3.9, in the specific case on a cubic non-linearity. It is the only part of the argument which makes use of the fact that u is a scalar field and not vector valued. The rest of the proof would go through in the vector-valued case and we expect that it is possible to find a vector-valued replacement for Lemma 2.6 as well.

In order to close the estimate obtained from Lemma 3.2.7, we require a bound that controls high order regularity of v in terms of the L^∞ norm. The classical method would consist of using a Schauder estimate of the form [49, Theorem 8.9.2]

$$[u]_{\delta+2, D_R} R^2 \lesssim [(\partial_t - \Delta)u + u]_{\delta, D}$$

for solutions of the inhomogeneous heat equation. Then if the right-hand side depends on lower order norms of u it can be absorbed into the left-hand side. We perform such an argument in the case where usual Hölder norms are replaced by the norms of "modelled distributions" (which depend on the underlying noise ζ).

First, power counting suggests that $v + \varUpsilon$ has better regularity than v (namely $1 - 2\epsilon$) and that this would be enough to define $v^2\varXi = v(v + \varUpsilon)\varXi - \varUpsilon\varXi$ (assuming that we can construct $\varUpsilon\varXi$), but not enough to define $v\mathbb{V}$. The next idea to get even better description of solution by explicit stochastic terms is to freeze coefficients at base point, and to look at local expansions that depend on that base point. The expansion of v in around base point x goes as follows:

$$v(y) = v(x) - (\varUpsilon(y) - \varUpsilon(x)) - 3v(x)(\mathbb{V}(y) - \mathbb{V}(x)). \tag{3.2.26}$$

We introduce the following function of two variables based on this local description:

$$U(y,x) = v(y) - v(x) + \varUpsilon(y) - \varUpsilon(x) + 3v(x)(\mathbb{V}(y) - \mathbb{V}(x)). \tag{3.2.27}$$

The regularity of U, as defined in (3.2.4), is expected to be higher than 1. This better description is indeed enough to define $v\varv$. The core observation is the following abstract reconstruction theorem, which is a variant of [41, Theorem 3.10] and [63, Proposition 1].

Theorem 3.2.8 (Reconstruction)**.** *Let $\gamma > 0$ and A be a finite subset of $(-\infty, \gamma]$. Let $L \in (0,1)$ and $x \in \mathbb{R} \times \mathbb{R}^3$. For a function $F\colon B(x,L)^2 \to \mathbb{R}$ assume that for all $\beta \in A$ there exist constants $C_\beta > 0$ and $\gamma_\beta \geqslant \gamma$ such that for all $t \in (0, L)$, for all $x_1, x_2 \in B(x, L-t)$*

$$\Big| \int \Psi_t(x_2 - y)(F(y, x_1) - F(y, x_2))dy \Big| \leqslant \sum_{\beta \in A} C_\beta d(x_1, x_2)^{\gamma_\beta - \beta} t^\beta. \tag{3.2.28}$$

Then $f : y \mapsto F(y,y)$ satisfies

$$\Big| \int \Psi_L(x - y)(F(y, x) - f(y))dy \Big| \lesssim \sum_{\beta \in A} C_\beta L^{\gamma_\beta}, \tag{3.2.29}$$

where "$\lesssim$" represents a bound up to a multiplicative constant depending only on γ and A.

As a consequence of this theorem, we get the following bounds on the products.

Lemma 3.2.9. *The following bound on $v^2\varXi$ holds:*

$$\begin{aligned} |(v^2\varXi)_L(x)| \lesssim & L^{\frac{1}{2}-3\epsilon}\|v\|_{B(x,L)}[v + \varUpsilon]_{1-2\epsilon, B(x,L)}[\varXi]_{-\frac{1}{2}-\epsilon} \\ & + L^{\frac{1}{2}-7\epsilon}[v]_{\frac{1}{2}-3\epsilon, B(x,L)}([\varUpsilon]_{\frac{1}{2}-3\epsilon}[\varXi]_{-\frac{1}{2}-\epsilon} + [\varUpsilon]_{-4\epsilon}) \\ & + \|v\|^2_{B(x,L)}[\varXi]_{-\frac{1}{2}-\epsilon}L^{-\frac{1}{2}-\epsilon} + \|v\|_{B(x,L)}[\varUpsilon]_{-4\epsilon}L^{-4\epsilon}. \end{aligned} \tag{3.2.30}$$

Lemma 3.2.10. *The following bound on $v\varv$ holds:*

$$\begin{aligned} & |((v - v(x))\varv)_L(x) + 3C_2(v_L + \varXi_L)(x)| \lesssim \\ & L^{\frac{1}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon, B(x,L)}[\varUpsilon]_{-4\epsilon} + [U]_{\frac{3}{2}-5\epsilon, B(x,L)}[\varv]_{-1-2\epsilon} + [\nu]_{\frac{1}{2}-5\epsilon, B(x,L)}[\varv_x]_{-2\epsilon}\Big) \\ & \quad + [\varUpsilon]_{-\frac{1}{2}-5\epsilon}L^{-\frac{1}{2}-5\epsilon} + L^{-4\epsilon}\|v\|_{B(x,L)}[\varUpsilon]_{-4\epsilon} + \|\nu\|_{B(x,L)}[\varv_x]_{-2\epsilon}L^{-2\epsilon}, \end{aligned} \tag{3.2.31}$$

where U is as introduced in (3.2.27) *and ν is optimal in the Definition 3.2.4.*

The reconstruction lemma and the two bounds above are proved in section 3.5.1 To bound the quantities appearing in the right-hand side of these lemmas, we will introduce our version of Schauder theory in the next section.

3.2.4 Schauder Theory

Here we present Schauder theory adapted to our set-up: regularity is measured by convolution with a kernel and we are interested in functions depending on base points. This is inspired from [63, Proposition 2], our contribution being the introduction of blow-up at the boundaries of the domain instead of an assumption of periodicity.

The formulation of the lemmas differs depending on the regularity, and we actually only use Lemma 3.2.3 in this chapter. All lemmas are used in the next chapter however and the proofs are simple rewritings of each other, so we group all of the statements here. The proof is in section 3.5.2.

In all this section, "$\lesssim$" denotes a bound that holds up to a multiplicative constant that only depends on κ and A when relevant.

Lemma 3.2.1. *Let $\kappa < 0$ and U be a distribution such that $(\partial_t - \Delta)U$ is compactly supported and*

$$\|(\partial_t - \Delta)U_L\| \leqslant ML^{\kappa-2}. \tag{3.2.32}$$

Then

$$[U]_\kappa \lesssim M. \tag{3.2.33}$$

The second lemma is slightly non-standard due to the presence of a second argument

Lemma 3.2.2. *Let $0 < \kappa < 2$ and $A \subset (-\infty, \kappa]$ be finite. Let U be a function of two variables such that $U(x,x) = 0$ for all x and $(\partial_t - \Delta)U(\cdot, x)$ is compactly supported for all x. Assume that there exists a constant $M^{(1)}$ such that for all base-points x and length scales $L_2 \leqslant L_1 \leqslant 1$, it holds that*

$$L_2^2 \|(\partial_t - \Delta)U_{L_2}(\cdot, x)\|_{B(x,L_1)} \leqslant M^{(1)} \sum_{\beta \in A} L_2^\beta {L_1}^{\kappa-\beta}. \tag{3.2.34}$$

Assume furthermore that there exists a constant $M^{(2)}$ such that, for any $x, y \in \mathbb{R} \times \mathbb{R}^d$, there exists $\lambda(y,x) = (\lambda^{(i)}(y,x))_{i=1}^d \in \mathbb{R}^d$ such that, for any $z \in \mathbb{R} \times \mathbb{R}^d$, the following "three-point continuity" holds:

$$|U(z,x) - U(y,x) - U(z,y) - \lambda(y,x) \cdot X(z-y)| \leqslant M^{(2)} \sum_{\beta \in A} d(y,x)^\beta d(z,y)^{\kappa-\beta}. \tag{3.2.35}$$

Note that if $\kappa \leqslant 1$ then λ doesn't matter so we can take $\lambda = 0$. Then

$$[U]_\kappa \lesssim M^{(1)} + M^{(2)}. \tag{3.2.36}$$

We now introduce the localised lemma that we use to bound the solutions. The difference

between this lemma and the previous one is that instead of a compact support of the right-hand side, we introduce a blow-up at the boundary in the way we measure objects.

Lemma 3.2.3. *Let $1 < \kappa < 2$ and $A \subset (-\infty, \kappa]$ be finite. Let U be a bounded function of two variables defined on a domain $D \times D$ such that $U(x,x) = 0$ for all x. Let $d_0 > 0$ and assume that for any $0 < d \leqslant d_0$ and $L_1 \leqslant \frac{d}{4}$ there exists a constant $M^{(1)}_{D_d, L_1}$ such that for all base-points $x \in D_d$ and length scales $L_2 \leqslant L_1$, it holds that*

$$L_2^2 \|(\partial_t - \Delta) U_{L_2}(\cdot, x)\|_{B(x, L_1)} \leqslant M^{(1)}_{D_d, L_1} \sum_{\beta \in A} L_2^{\beta} {L_1}^{\kappa - \beta}. \tag{3.2.37}$$

Assume furthermore, that for $L_1, L_2 \leqslant \frac{d}{4}$ there exists a constant $M^{(2)}_{D_d, L_1, L_2}$ such that, for any $x \in D_d$ and $y \in B(x, L_1)$, there exists $\lambda(y,x) = (\lambda^{(i)}(y,x))_{i=1}^d \in \mathbb{R}^d$ such that, for any $z \in B(y, L_2)$, the following "three-point continuity" holds:

$$\begin{aligned} |U(z,x) - U(y,x) - U(z,y) - \lambda(y,x) \cdot X(z-y)| \\ \leqslant M^{(2)}_{D_d, L_1, L_2} \sum_{\beta \in A} d(y,x)^{\beta} d(z,y)^{\kappa - \beta}. \end{aligned} \tag{3.2.38}$$

Additionally define

$$M^{(1)} := \sup_{d \leqslant d_0} d^{\kappa} M^{(1)}_{D_d, \frac{d}{2}}, \quad \text{and} \quad M^{(2)} := \sup_{d \leqslant d_0} d^{\kappa} M^{(2)}_{D_d, \frac{d}{2}, \frac{d}{4}}.$$

Then

$$\sup_{d \leqslant d_0} d^{\kappa} [U]_{\kappa, D_d} \lesssim M^{(1)} + M^{(2)} + \sup_{d \leqslant d_0} \|U\|_{D_d, d}. \tag{3.2.39}$$

The following lemma gives bounds on the derivative. It can be used both for the derivatives of the trees in Section 4.4 and for the derivative of the solution in Section 4.9.

Lemma 3.2.4. *Let $\kappa > 1$ and $U \in C^{\kappa}(\mathbb{R} \times \mathbb{R}^d)$ then, for the optimal function ν in* (3.2.4)*, for any $r \in (0, \infty)$,*

$$\|\nu\| \lesssim r^{\kappa - 1} [U]_{\kappa} + r^{-1} \|U\| \,. \tag{3.2.40}$$

Suppose furthermore that there exists a constant M and, for all $x, y \in \mathbb{R} \times \mathbb{R}^d$, a vector $\lambda(y,x) = (\lambda^{(i)}(y,x))_{i=1}^d \in \mathbb{R}^d$ such that for any $z \in \mathbb{R} \times \mathbb{R}^d$ one has the three-point continuity bound

$$|U(z,x) - U(y,x) - U(z,y) - \lambda(y,x) \cdot X(z-y)| \leqslant M \sum_{\beta \in A} d(y,x)^{\beta} d(z,y)^{\kappa - \beta} \,. \tag{3.2.41}$$

Then, if we write $f(z,w) = (f^{(i)}(z,w))_{i=1}^d$ where $f^{(i)}(z,w) = \nu^{(i)}(z) - \nu^{(i)}(w) +$

$\lambda^{(i)}(z,w)$, *one has*

$$[f]_{\kappa-1} \lesssim [U]_\kappa + M \ . \tag{3.2.42}$$

A localised version of this lemma is as follows:

Lemma 3.2.5. *Assume that D satisfies a spatial interior cone condition with parameters $r_0 > 0$ and $\beta \in (0,1)$, i.e. for all $r \in [0, r_0]$, for all $x \in D$, for any vector $\theta = (\theta^{(i)})_{i=1}^d \in \mathbb{R}^d$, there exists $y \in D$ such that $d(x,y) = r$ and*

$$|\theta^{(i)} y_i| \geqslant \beta d(x,y)|\theta|.$$

Let $\kappa > 1$ and $U \in C^\kappa$ then, for the optimal function ν in (3.2.4) *and for all $r \in [0, r_0]$, we have the bound*

$$\beta \|\nu\|_D \leqslant r^{\kappa-1}[U]_{\kappa,D} + r^{-1}\|U\|_{D,r} \ . \tag{3.2.43}$$

Suppose furthermore that there exists a constant M and, for all $x, y \in D$, a vector $\lambda(y,x) = (\lambda^{(i)}(y,x))_{i=1}^d \in \mathbb{R}^d$ such that for any $z \in D$ one has the the three-point continuity bound

$$|U(z,x) - U(y,x) - U(z,y) - \lambda(y,x)\cdot X(z-y)| \leqslant M \sum_{\beta \in A} d(y,x)^\beta d(z,y)^{\kappa-\beta} \ . \tag{3.2.44}$$

Then, if we write $f(z,w) = (f^{(i)}(z,w))_{i=1}^d$ where $f^{(i)}(z,w) = \nu^{(i)}(z) - \nu^{(i)}(w) + \lambda^{(i)}(z,w)$, one has, for every $r \in [0, r_0]$,

$$[f]_{\kappa-1,D} \lesssim [U]_{\kappa,D} + M + r^{-\kappa}\|U\|_{D,r} \ . \tag{3.2.45}$$

3.3 Translation to the language of regularity structures

Although our argument is not formulated using the terminology of the theory of regularity structures, the analysis of the small scale behaviour, Theorem 3.2.8, Lemmas 3.2.9 and 3.2.10 as well as the Schauder estimate 3.2.3, build on the same key ideas as this theory. We now provide a translation of how the lemmas that appear in this section can be stated in terms of the central objects introduced in the theory of regularity structures such as the models, modelled distributions, and the abstract integration operator. This section is aimed at those interested in this theory but plays no role in the results presented in this chapter, beyond explaining why the assumptions we make in section 3.2.2 are reasonable in this framework. In the next chapter however we will revisit the theory of regularity structures and introduce a slightly different formulation.

We begin by recalling the setup of regularity structures, introduced in section 1.2: in [41, Definition 2.1] a regularity structure is defined as a triple (A, T, G), consisting of

an index set $A \subset \mathbb{R}$, a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$ and a group G of linear transformations acting on T with some additional properties. In this framework, the local description of the solution u is encoded by replacing the scalar valued function / distribution u by modelled distribution, which is a function $\mathcal{U} : \mathbb{R} \times \mathbb{R}^3 \to T$ for a certain purpose-built regularity structure. To build this structure one first introduces some symbols, namely

$$\{\overline{\mathbf{1}}, \overline{\mathbf{I}}, \overline{\mathbf{\Psi}}, \overline{\mathbf{Y}}\} \cup \{\overline{X_i} : i = 1, 2, 3\}.$$

At this level these symbols with a bar on top are completely abstract objects, but of course they ultimately represent the functions / distributions appearing in the local description. To each of these symbols τ one associates a *homogeneity* $|\tau| \in \mathbb{R}$, namely

$$|\overline{\mathbf{1}}| = 0, \quad |\overline{X_i}| = 1, \quad |\overline{\mathbf{I}}| = -\frac{1}{2} - \epsilon, \quad |\overline{\mathbf{\Psi}}| = \frac{1}{2} - 3\epsilon, \quad |\overline{\mathbf{Y}}| = 1 - 2\epsilon.$$

The space T is then defined as the finite dimensional space

$$T = \bigoplus_{\tau \in \{\overline{\mathbf{1}}, \overline{X_i}, \overline{\mathbf{I}}, \overline{\mathbf{\Psi}}, \overline{\mathbf{Y}}\}} \mathbb{R}\tau,$$

and A is defined to be the set of homogeneities of these symbols. It turns out that the modelled distribution $\mathcal{U}$ takes the form

$$\mathcal{U}(x) = \overline{\mathbf{I}} + v(x)\overline{\mathbf{1}} - \overline{\mathbf{\Psi}} - 3v(x)\overline{\mathbf{Y}} - \nu(x).\overline{X}, \tag{3.3.1}$$

for some functions v and ν (which of course coincide with our functions v and ν). For our analysis we choose to work with a local description for $v = u - \mathbf{I}$, which in the notation of regularity structures would take the form

$$\mathcal{V}(x) = v(x)\overline{\mathbf{1}} - \overline{\mathbf{\Psi}} - 3v(x)\overline{\mathbf{Y}} - \nu(x).\overline{X}, \tag{3.3.2}$$

i.e. the only difference with respect to (3.3.1) is that the term $\overline{\mathbf{I}}$ is removed. Equation (3.3.2) should be viewed as an abstract counterpart of our equation (3.2.26). For us it is more convenient to work with v rather than u to get good bounds on the error term $(v_T)^3 - (v^3)_T$. We argue below that the regularity assumption we impose on $\mathcal{V}$ is equivalent to the condition imposed on $\mathcal{U}$ in [41].

Just like our main result, Theorem 3.2.1, the solution theory using regularity structures requires a perturbative expansion as an input. There this expansion is encoded in the notion of a model [41, Definition 2.17]. To each of the symbols, one associates a function

or distribution $\mathbf{\Pi}\tau$ corresponding exactly to our definitions (3.2.11), and (3.2.13), i.e.

$$\mathbf{\Pi}\bar{\mathbf{1}}(y) = 1, \qquad \mathbf{\Pi}\overline{X_i}(y) = y_i, \qquad \mathbf{\Pi}\bar{\mathbf{l}}(y) = \mathbf{l}(y),$$
$$\mathbf{\Pi}\bar{\mathbf{\Psi}}(y) = \mathbf{\Psi}(y), \qquad \mathbf{\Pi}\bar{\Upsilon}(y) = \Upsilon(y).$$

A key idea of the theory is to not work with these distributions directly, but with *centred* or *positively renormalised* objects, $\Pi_x\tau$ indexed by a base-point in $x \in \mathbb{R} \times \mathbb{R}^3$. The right notion of regularity for the *modelled distributions* $\mathcal{U}$ and $\mathcal{V}$ is then defined in term of this recentring procedure.

For the symbols we have introduced so far, the centring is relatively simple and amounts to subtracting the value at the base point for the symbols of strictly positive homogeneity:

$$\Pi_x\bar{\mathbf{1}}(y) = 1, \qquad \Pi_x\overline{X_i}(y) = y_i - x_i, \qquad \Pi_x\bar{\mathbf{l}}(y) = \mathbf{l}(y),$$
$$\Pi_x\bar{\Upsilon}(y) = \Upsilon(y) - \Upsilon(x), \qquad \Pi_x\bar{\mathbf{\Psi}}(y) = \mathbf{\Psi}(y) - \mathbf{\Psi}(x).$$

The reason why one works with these centred objects is that one has good control over their behaviour as the argument approaches the base point x. This is encoded in the formula [41, Equation (2.15)],

$$\langle \Pi_x\tau, \varphi_x^\lambda \rangle \lesssim \lambda^{|\tau|}, \tag{3.3.3}$$

where φ is a smooth test-function rescaled to scale λ and centred at the base-point x. This corresponds exactly to our regularity assumption on the Hölder norms of the objects, see Section 3.2.2 (our scale is called L rather than λ and the test-function is called Ψ rather than φ).

In order to connect the centring procedure to the functions $\mathcal{U}$ and $\mathcal{V}$ and to formulate the right continuity condition, it is useful to introduce the structure group G. In the current context this group is simply the five-dimensional group of all linear transformations F on T of the form

$$F\bar{\mathbf{1}} = \bar{\mathbf{1}}, \qquad F\overline{X_i} = \overline{X_i} + a_i\bar{\mathbf{1}} \quad a_i \in \mathbb{R}, \qquad F\bar{\mathbf{l}} = \bar{\mathbf{l}},$$
$$F\bar{\mathbf{\Psi}} = \bar{\mathbf{\Psi}} + b\bar{\mathbf{1}} \quad b \in \mathbb{R}, \qquad F\bar{\Upsilon} = \bar{\Upsilon} + c\bar{\mathbf{1}} \quad c \in \mathbb{R}, \tag{3.3.4}$$

but this group will be enlarged as more symbols are introduced below. For each $x \in \mathbb{R} \times \mathbb{R}^3$ we define $F_x \in G$ by

$$F_x\bar{\mathbf{1}} = \bar{\mathbf{1}}, \qquad F_x\overline{X_i}(y) = \overline{X_i} - x_i\bar{\mathbf{1}}, \qquad F_x\bar{\mathbf{l}} = \bar{\mathbf{l}},$$
$$F_x\bar{\Upsilon} = \bar{\Upsilon} - \Upsilon(x)\bar{\mathbf{1}}, \qquad F_x\bar{\mathbf{\Psi}} = \bar{\mathbf{\Psi}} - \mathbf{\Psi}(x)\bar{\mathbf{1}},$$

so that one gets

$$\Pi_x \tau = \mathbf{\Pi} F_x \tau.$$

Now, for $x, y \in \mathbb{R} \times \mathbb{R}^3$ we set

$$\Gamma_{xy} = F_y^{-1} \circ F_x,$$

and we trivially have the identity, cf. [41, Definition 2.17].

$$\Pi_x = \Pi_y \circ \Gamma_{xy}. \tag{3.3.5}$$

The continuity assumption on $\mathcal{U}$ and $\mathcal{V}$ is formulated in terms of the translation operators Γ_{xy}. $\mathcal{U}$ is said to be a modelled distribution of order γ if

$$\|\mathcal{U}(x) - \Gamma_{xy}\mathcal{U}(y)\|_\beta \lesssim d(x,y)^{\gamma-\beta},$$

where $\|\cdot\|_\beta$ refers to the component in T_β. It is easy to check that for both, $\mathcal{U}$ defined by (3.3.1) and $\mathcal{V}$ defined by (3.3.2), this condition translates precisely into the "modelledness conditions"

$$\begin{aligned} |v(y) - v(x) + \mathbb{V}(y) - \mathbb{V}(x) - \nu(x).X(y-x) \\ +3v(x)(\mathbb{Y}(y) - \mathbb{Y}(x))| &\lesssim d(x,y)^\gamma, \\ |\nu(y) - \nu(x)| &\lesssim d(x,y)^{\gamma-1}, \\ |v(y) - v(x)| &\lesssim d(x,y)^{\gamma-1+2\varepsilon}, \end{aligned} \tag{3.3.6}$$

and this condition for $\gamma = \frac{3}{2} - 5\epsilon$ corresponds exactly to the regularity assumptions on $\mathcal{U}$, ν and v we work with.

The main feature of the space of modelled distributions is that although expansions like (3.3.1) are ultimately used as good local descriptions of distributions, one can multiply them as if they were of positive regularity, provided one can expand the action of the model to new symbols that are seen as products of the symbols introduced earlier. For equation (3.1.1) one has to get a bound on $u^3 = (v + \mathbb{I})^3 = v^3 + 3v^2\mathbb{I} + 3v\mathbb{V} + \mathbb{V}$. We aim to bound this in terms of:

- A high-regularity norm on v, namely the $\mathcal{D}^\gamma$ norm of the modelled distribution $\mathcal{V}$, which is defined as the smallest possible constant in the inequalities (3.3.6);
- The low regularity L^∞ norm $\|v\|$;
- The bounds on the various stochastic terms.

The term v^3 can immediately be bounded by $\|v\|^3$ and $\mathbb{V}$ is a stochastic term which does

not involve v. The only terms which require work are $3v^2\,\raisebox{0.2em}{\scriptsize ı}$ and $3v\,\raisebox{0.2em}{\scriptsize V}$. The distribution $\raisebox{0.2em}{\scriptsize V}$ has regularity $-1-2\varepsilon$, so a description of v to order $\gamma > 1+2\varepsilon$ is required. Such a description is precisely provided by (3.3.2). One now defines new symbols

$$\{\overline{\raisebox{0.2em}{\scriptsize V}},\ \overline{\raisebox{0.2em}{\scriptsize V}},\ \overline{\raisebox{0.2em}{\scriptsize V}},\ \overline{\raisebox{0.2em}{\scriptsize V}},\ \overline{\raisebox{0.2em}{\scriptsize V}}\}, \tag{3.3.7}$$

associates to them a homogeneity using the rule $|\tau\overline{\tau}| = |\tau| + |\overline{\tau}|$, and simply defines a new modelled distribution for the local description of $v\,\raisebox{0.2em}{\scriptsize V}$ by

$$\mathcal{V}\,\raisebox{0.2em}{\scriptsize V}(x) = v(x)\overline{\raisebox{0.2em}{\scriptsize V}} + \overline{\raisebox{0.2em}{\scriptsize V}} - 3v(x)\overline{\raisebox{0.2em}{\scriptsize V}} - \nu(x).\overline{\raisebox{0.2em}{\scriptsize V}}. \tag{3.3.8}$$

This definition becomes substantial by extending the model (Π_x, Γ_{xy}) to these new symbols. One would like to extend the operator Π_x to these products simply by defining locally

$$\Pi_x(\tau\overline{\tau})(y) = (\Pi_x\tau(y))(\Pi_x\overline{\tau}(y)),$$

but such a definition may not be meaningful when the regularisation is removed. Fortunately, there is some flexibility at this level. The main requirements for multiplication to be well-behaved are only that (3.3.3) and (3.3.5) remain valid for the new symbols and additionally that one has the identity

$$\Gamma_{xy}(\tau\overline{\tau}) = (\Gamma_{xy}\tau)(\Gamma_{xy}\overline{\tau}).$$

It is here that the positive renormalisation, and hence the action of the structure group G becomes more involved than subtracting the value at a base point and the condition $|\tau\overline{\tau}| = |\tau| + |\overline{\tau}|$ becomes strictly stronger than Hölder regularity. For example $\mathbf{\Pi}\overline{\raisebox{0.2em}{\scriptsize V}}$ is a distribution of regularity $-1-2\epsilon$ but its homogeneity is strictly larger, namely $-\frac{1}{2}-5\epsilon$. The condition (3.3.3) states that near any base point x, $\mathbf{\Pi}\overline{\raisebox{0.2em}{\scriptsize V}}$ is well described by a $\raisebox{0.2em}{\scriptsize V}(x)\mathbf{\Pi}\overline{\raisebox{0.2em}{\scriptsize V}}$ up to an error of order $-\frac{1}{2}-5\epsilon$, which is strictly stronger than a bound on the $C^{-1-2\epsilon}$ norm of it. Our definitions (3.2.12) and the assumed bounds (3.2.17), (3.2.15), (3.2.14) correspond exactly to the definitions for Π_x and the bound (3.3.3) in [41]. The only difference is that Hairer defines the trees $\raisebox{0.2em}{\scriptsize ı}$,$\raisebox{0.2em}{\scriptsize V}$ and $\raisebox{0.2em}{\scriptsize V}$ using the inverse heat operator with some cutoff at large scales and appropriate right-hand side. We only assume that they satisfy the heat equation point-wise without imposing any boundary conditions, but we additionally impose some natural regularity bounds, as explained in Remark 3.2.5. Combining Hairer's multiplication theorem [41, Theorem 4.7] and his reconstruction theorem [41, Theorem 3.10] then yields the estimate

$$\big|\big\langle \mathcal{R}(\mathcal{V}\,\raisebox{0.2em}{\scriptsize V}) - \Pi_x(\mathcal{V}\,\raisebox{0.2em}{\scriptsize V}), \varphi_x^\lambda\big\rangle\big| \lesssim \lambda^{\gamma-1-2\varepsilon}\|\mathcal{V}\|_{\mathcal{D}^\gamma}\|\Pi\|,$$

where $\|\Pi\|$ is the smallest possible constant in all of the assumed bounds on the model. This is essentially the statement of our Lemma 3.2.10 up to a few points:

- Some of the terms in $\Pi_x(\mathcal{V}\Xi)$ are removed from the left-hand side of (3.2.31) and added to the right-hand side using the triangle inequality.
- We prove these estimates "by hand" without using the algebraic machinery discussed above and in particular without introducing the group G to organise the various continuity assumptions. More precisely, Theorem 3.2.8, is a condensed version of [41, Theorem 3.10] which contains the key analytic estimate, but assumes the output of the algebraic machinery. In the case, of (3.1.1) the algebraic manipulations are not too complex and can be done directly quite easily, and that is precisely what we do in the proof of Lemma 3.2.10
- Along the way we keep track of the precise norms needed in each term, rather than compiling them in $\|\mathcal{V}\|_{\mathcal{D}^\gamma}$ and $\|\Pi\|$. This added level of detail is important for us, especially when determining the exact exponents of each tree appearing in our final estimate (3.2.19).

The treatment of the term $v^2\Xi$ goes along similar lines. As Ξ has better regularity than Ξ, namely $-\frac{1}{2}-\varepsilon$, a local description of v^2 is only required to order $> \frac{1}{2}+\varepsilon$ and this is provided by

$$\mathcal{V}^2(x) = v^2(x)\overline{\mathbf{1}} - 2v(x)\overline{\Xi},$$

which in turn prompts us to define

$$\mathcal{V}^2\Xi(x) = v^2(x)\overline{\Xi} - 2v(x)\overline{\Xi}.$$

Again, our assumption (3.2.16) corresponds exactly to the homogeneity condition (3.3.3) in [41] and our Lemma 3.2.9 is obtained by combining the multiplication and reconstruction theorem and applying the triangle inequality, this time to remove the term corresponding to $\Pi_x\mathcal{V}^2\Xi$ from the left hand side completely. The Hölder norm $[v+\Xi]_{1-2\epsilon}$ which appears on the right hand side of our estimate (3.2.30) corresponds to the norm of the modelled distribution one obtains by removing the terms $-3v(x)\overline{\Xi} - \nu(x).\overline{X}$, which are not necessary here, from the definition of $\mathcal{V}$ in (3.3.2).

The last ingredient from the theory of regularity structures concerns the heat operator. For us, the gain of regularity for solutions to the heat equation is expressed in Lemma 3.2.3, and this corresponds to [41, Theorem 5.12]. As stated above in (3.3.6), we seek a local description of the solution v of order $\gamma = \frac{3}{2} - 5\epsilon$. The heat operator $(\partial_t - \Delta)^{-1}$ is a 2-regularising operator ($\beta = 2$ in Hairer's theory) and thus it seems reasonable that a local description of the right-hand side of (3.2.18) up to order $-\frac{1}{2} - 5\epsilon$ is required as

input for the Schauder Lemma. Therefore we work with

$$\mathcal{W}(x) = v(x)\overline{\mathbf{V}} + \overline{\mathbf{V}}.$$

At this point we slightly deviate from Hairer's approach: his Schauder Lemma [41, Theorem 5.12] assumes that the right hand side is a modelled distribution of strictly positive order, because he applies the reconstruction operator as part of the argument. We circumvent this by viewing the reconstruction as an extra input to the theorem. Imposing that $\mathcal{W}$ is a modelled distribution of order $-\frac{1}{2}-5\epsilon$ [41, Definition 3.1] translates precisely to our three-point continuity condition (3.2.38), and our smallness assumption (3.2.37) corresponds to [41, Equation 5.42], which in the notation of this section would be

$$\left|\langle \mathcal{R}\mathcal{W} - \Pi_x \mathcal{W}, \varphi_x^\lambda \rangle\right| \lesssim \lambda^{-\frac{1}{2}-5\epsilon}. \tag{3.3.9}$$

The exact statement of the assumption (3.2.38) appears slightly stronger than (3.3.9) because of the L^∞ norm on the left-hand side and the extra parameter L, but in practice the seemingly stronger bound can be obtained easily from the weaker bound using triangle inequality and some lower-order regularity information.

In the framework of regularity structures the operator that encodes the integration of a modelled distribution is described as the sum of three operators. The first operator $\mathcal{I}$ acts point-wise on the modelled distributions by a shift of coefficients. The action on the trees is:

$$\mathcal{I}\overline{\mathbf{V}} = \overline{\mathbf{V}}, \quad \mathcal{I}\overline{\mathbf{V}} = \overline{\mathbf{V}}.$$

The continuity of the coefficients for a modelled distribution is transferred accordingly under the action of $\mathcal{I}$. In our setting, this is also automatic and follows from our assumptions, as explained in Remark 3.2.5.

The non-trivial part of the integration happens on the levels $\overline{\mathbf{1}}$ and $\overline{X}$ which is encoded in Hairer's theory in the operators $\mathcal{J}$ and $\mathcal{N}$. We have again a direct translation, although we do not need to split the operator.

$$\begin{aligned}
\mathcal{N}\mathcal{W}(x) &= \big((\partial_t - \Delta)^{-1}(-v^3 - 3v^2\mathbf{V} - 3(v - v(x))\mathbf{V})|_{y=x}\big)\overline{\mathbf{1}} + \nu(x)\overline{X}, \\
\mathcal{J}(x)\mathcal{W} &= \big(3v(x)\mathbf{V} - \mathbf{V}\big)\overline{\mathbf{1}}.
\end{aligned}$$

The differences between our approach and the one adopted by Hairer is that in the spirit of [62] we use a kernel-free approach and we have a special treatment of the boundary on the levels $\overline{\mathbf{1}}$ and $\overline{X}$. We are also more precise in our final bounds in the sense that, as in the definition of $\mathcal{V}\mathbf{V}$, we keep track of the precise norms needed in each term.

3.4 Proof of Theorem 3.2.1

3.4.1 Assumption

We assume that the bound of Theorem 3.2.1 in terms of powers of trees does not hold on a domain $D = P_r$, and use that assumption to prove that then bound in $\frac{1}{R}$ holds[1]. Our assumption is stated as such:

$$\forall \tau \in \mathcal{T}, \quad [\tau]_{\gamma_\tau} \leqslant c\|v\|_D^{n_\tau(\frac{1}{2}-\epsilon)}, \tag{3.4.1}$$

for some constant $c < 1$ that we will tune later, according to conditions suggested by equations (3.4.18) and (3.4.32). With these assumptions, Lemmas 3.2.9 and 3.2.10 can be restated as, for any x with $B(x, L) \in D$,

$$\begin{aligned}|(v^2\,\raisebox{-0.2ex}{\scalebox{1.2}{\textbullet}})_L(x)| \lesssim & c([v + \Upsilon]_{1-2\epsilon,B(x,L)} L^{\frac{1}{2}-3\epsilon}\|v\|_D^{\frac{3}{2}-\epsilon} + [v]_{\frac{1}{2}-3\epsilon,B(x,L)} L^{\frac{1}{2}-7\epsilon}\|v\|_D^{2-4\epsilon}) \\ & + c(L^{-\frac{1}{2}-\epsilon}\|v\|_D^{\frac{5}{2}-\epsilon} + L^{-4\epsilon}\|v\|_D^{3-4\epsilon} + L^{\frac{1}{2}-7\epsilon}\|v\|_D^{\frac{7}{2}-7\epsilon}).\end{aligned} \tag{3.4.2}$$

and:

$$\begin{aligned}&|((v - v(x))\mathbb{V})_L(x) + 3C_2(v_L + \,\raisebox{-0.2ex}{\scalebox{1.2}{\textbullet}}_L)(x)| \\ &\lesssim cL^{\frac{1}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon,B(x,L)}\|v\|_D^{2-4\epsilon} + ([U]_{\frac{3}{2}-5\epsilon,B(x,L)} + [\nu]_{\frac{1}{2}-5\epsilon,B(x,L)})\|v\|_D^{1-2\epsilon}\Big) \\ &\quad + c(\|v\|_D^{\frac{5}{2}-5\epsilon}L^{-\frac{1}{2}-5\epsilon} + L^{-4\epsilon}\|v\|_D^{3-4\epsilon}) + \|\nu\|_{B(x,L)}c\|v\|_D^{1-2\epsilon}L^{-2\epsilon}.\end{aligned} \tag{3.4.3}$$

3.4.2 Applying Lemma 3.2.3

For any domain D, for x, L, T and d such that $B(x, L + T) \subset D_d$, we prove the following bound, which is the first condition to apply Lemma 3.2.3.

$$\begin{aligned}\|(\partial_t - \Delta)U(x,\cdot)_T\|_{B(x,L)} \lesssim & \|v\|_D^3 + L^{\frac{1}{2}-3\epsilon}T^{-1-2\epsilon}c[v]_{\frac{1}{2}-3\epsilon,D_d,d}\|v\|_D^{1-2\epsilon} \\ & + c([v + \Upsilon]_{1-2\epsilon,B(x,T)}T^{\frac{1}{2}-3\epsilon}\|v\|_D^{\frac{3}{2}-\epsilon} + [v]_{\frac{1}{2}-3\epsilon,B(x,T)}T^{\frac{1}{2}-7\epsilon}\|v\|_D^{2-4\epsilon}) \\ & + c(T^{-\frac{1}{2}-\epsilon}\|v\|_D^{\frac{5}{2}-\epsilon} + T^{-4\epsilon}\|v\|_D^{3-4\epsilon} + T^{\frac{1}{2}-7\epsilon}\|v\|_D^{\frac{7}{2}-7\epsilon}) \\ & + cT^{\frac{1}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon,B(x,T)}\|v\|_D^{2-4\epsilon} + ([U]_{\frac{3}{2}-5\epsilon,B(x,T)} + [\nu]_{\frac{1}{2}-5\epsilon,B(x,T)})\|v\|_D^{1-2\epsilon}\Big) \\ & + c(\|v\|_D^{\frac{5}{2}-5\epsilon}T^{-\frac{1}{2}-5\epsilon} + T^{-4\epsilon}\|v\|_D^{3-4\epsilon}) + \|\nu\|_{B(x,T)}c\|v\|_D^{1-2\epsilon}T^{-2\epsilon},\end{aligned} \tag{3.4.4}$$

where ν is the optimal function in the definition of $[U]_{\frac{3}{2}-5\epsilon,D_d}$.

[1] This assumption is not mandatory to the proof but it simplifies greatly the computations, in particular (3.4.20), by allowing one to replace all occurrences of norms of trees by powers of $\|v\|_D$.

Let x be an arbitrary point in D_{d+L} and y a point in $B(x,L) \subset D_d$. We have

$$\begin{aligned}(\partial_t - \Delta)U(x,\cdot)_T(y) &= \int \Psi_T(z-y)(\partial_t - \Delta)U(x,z)dz \\ &= -(v^3)_T(y) - 3(v^2\mathring{\mathbb{V}})_T(y) - 3((v-v(x))\mathbb{V})_T(y) - 9C_2(v_T(y) + \mathring{\mathbb{V}}_T(y)) \\ &= -(v^3)_T(y) - 3(v^2\mathring{\mathbb{V}})_T(y) - 3(v(y)-v(x))\mathbb{V}_T(y) \\ &\quad -3((v-v(y))\mathbb{V})_T(y) - 9C_2(v_T(y) + \mathring{\mathbb{V}}_T(y)).\end{aligned}$$

We bound the some terms of this expression by the previous bounds (3.4.2) and (3.4.3) and the remaining ones as follows:

$$|(v^3)_T(y)| \leqslant \|v\|^3_{B(y,T)} \leqslant \|v\|^3_{D_d}, \tag{3.4.5}$$

$$\begin{aligned}|(v(y)-v(x))\mathbb{V}_T(y)| &\leqslant d(x,y)^{\frac{1}{2}-3\epsilon}[v]_{\frac{1}{2}-3\epsilon,D_d,d}T^{-1-2\epsilon}[\mathbb{V}]_{-1-2\epsilon} \\ &\leqslant L^{\frac{1}{2}-3\epsilon}T^{-1-2\epsilon}c[v]_{\frac{1}{2}-3\epsilon,D_d,d}\|v\|_D^{1-2\epsilon}\end{aligned} \tag{3.4.6}$$

This proves (3.4.4).

The three-point continuity on U holds as follows. For any $x \in D_d$, for any $y \in B(x, \frac{d}{4})$, for any $z \in B(y, \frac{d}{4})$

$$\begin{aligned}|U(y,x) - U(x,z) - U(z,y)| =& 3|v(x)-v(z)||\mathring{\mathbb{Y}}(y) - \mathring{\mathbb{Y}}(z)| \\ \leqslant& 3[v]_{\frac{1}{2}-3\epsilon,D_{\frac{d}{2}},\frac{d}{2}}[\mathring{\mathbb{Y}}]_{1-2\epsilon}d(x,z)^{\frac{1}{2}-3\epsilon}d(y,z)^{1-2\epsilon} \\ \leqslant& 3c[v]_{\frac{1}{2}-3\epsilon,D_{\frac{d}{2}},\frac{d}{2}}\|v\|_D^{1-2\epsilon}d(x,z)^{\frac{1}{2}-3\epsilon}d(y,z)^{1-2\epsilon}\end{aligned} \tag{3.4.7}$$

Lemma 3.2.3 applies to U with $\kappa = \frac{3}{2} - 5\epsilon$. Note that in the bound 3.4.4 we see powers of T higher than $T^{-\frac{1}{2}-5\epsilon}$ but we use the fact that $T \lesssim d$ to make up for that. After a few simplifications we get

$$\begin{aligned}\sup_{d\leqslant d_0} d^{\frac{3}{2}-5\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} \lesssim \sup_{d\leqslant d_0}\Big(& d^2\|v\|_D^3 + d^{\frac{3}{2}-5\epsilon}c[v]_{\frac{1}{2}-3\epsilon,D_d,d}\|v\|_D^{1-2\epsilon} \\ &+ c([v+\mathring{\mathbb{Y}}]_{1-2\epsilon,D_d,d}d^{\frac{5}{2}-3\epsilon}\|v\|_D^{\frac{3}{2}-\epsilon} + d^{\frac{3}{2}-\epsilon}\|v\|_D^{\frac{5}{2}-\epsilon} + d^{\frac{5}{2}-7\epsilon}\|v\|_D^{\frac{7}{2}-7\epsilon}) \\ &+ cd^{\frac{5}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon,D_d,d}\|v\|_D^{2-4\epsilon} + ([U]_{\frac{3}{2}-5\epsilon,D_d,d} + [\nu]_{\frac{1}{2}-5\epsilon,D_d,d})\|v\|_D^{1-2\epsilon}\Big) \\ &+ c(\|v\|_D^{\frac{5}{2}-5\epsilon}d^{\frac{3}{2}-5\epsilon} + d^{2-4\epsilon}\|v\|_D^{3-4\epsilon}) + \|\nu\|_{D_d}c\|v\|_D^{1-2\epsilon}d^{2-2\epsilon} + \|U\|_{D_d,d}\Big).\end{aligned} \tag{3.4.8}$$

3.4.3 Simplifications

Our goal in this section is to produce bounds on the norms $[v]_{\frac{1}{2}-3\epsilon,D_d,d}$, $[v+\Upsilon]_{1-2\epsilon,D_d,d}$ and $[U]_{\frac{3}{2}-5\epsilon,D_d}$ that depend only on $\|v\|_D$, in particular independent of each other. We introduce the following elementary bounds, which can be deduced from triangle inequalities and application of the Assumption 3.4.1.

$$\begin{aligned}[v]_{\frac{1}{2}-3\epsilon,D_d,d} &\leqslant d^{1-2\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + [\Upsilon]_{\frac{1}{2}-3\epsilon} + 3d^{\frac{1}{2}+\epsilon}\|v\|_{D_d}[\Upsilon]_{1-2\epsilon} + d^{\frac{1}{2}+3\epsilon}\|\nu\|_{D_d} \\ &\leqslant d^{1-2\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + c\|v\|_D^{\frac{3}{2}-3\epsilon} + 3cd^{\frac{1}{2}+\epsilon}\|v\|_{D_d}^{2-2\epsilon} + d^{\frac{1}{2}+3\epsilon}\|\nu\|_{D_d}\end{aligned} \tag{3.4.9}$$

and

$$\begin{aligned}[v+\Upsilon]_{1-2\epsilon,D_d,d} &\leqslant d^{\frac{1}{2}-3\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + 3\|v\|_{D_d}[\Upsilon]_{1-2\epsilon} + d^{2\epsilon}\|\nu\|_{D_d} \\ &\leqslant d^{\frac{1}{2}-3\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + 3c\|v\|_{D_d}^{2-2\epsilon} + d^{2\epsilon}\|\nu\|_{D_d},\end{aligned} \tag{3.4.10}$$

and we recall that from Corollary 3.2.5 we have the two bounds, assuming $d \in (0, r_d]$,

$$\|\nu\|_{D_d} \lesssim [U]_{\frac{3}{2}-5\epsilon,D_d} d^{\frac{1}{2}-5\epsilon} + \|U\|_{D_d,d} d^{-1} \tag{3.4.11}$$

and

$$\begin{aligned}[\nu]_{\frac{1}{2}-5\epsilon,D_d} &\lesssim [U]_{\frac{3}{2}-5\epsilon,D_d} + [v]_{\frac{1}{2}-3\epsilon,D_d,d}[\Upsilon]_{1-2\epsilon} + d^{-\frac{3}{2}+5\epsilon}\|U\|_{D_d,d} \\ &\lesssim [U]_{\frac{3}{2}-5\epsilon,D_d} + c[v]_{\frac{1}{2}-3\epsilon,D_d,d}\|v\|_D^{1-2\epsilon} + d^{-\frac{3}{2}+5\epsilon}\|U\|_{D_d,d}.\end{aligned} \tag{3.4.12}$$

We will also be using the bound

$$\begin{aligned}\|U\|_{D_d,d} &\leqslant 2\|v\|_D + [\Upsilon]_{\frac{1}{2}-3\epsilon} d^{\frac{1}{2}-3\epsilon} + 3\|v\|_D[\Upsilon]_{1-2\epsilon} d^{1-2\epsilon} \\ &\leqslant 2\|v\|_D + cd^{\frac{1}{2}-3\epsilon}\|v\|_D^{\frac{3}{2}-3\epsilon} + 3cd^{1-2\epsilon}\|v\|_D^{2-2\epsilon}.\end{aligned} \tag{3.4.13}$$

By combining the bounds above to get bounds in terms of $\|v\|_D$ and $[U]_{\frac{3}{2}-5\epsilon,D_d}$ only, we get the following bounds (in order of logical deduction):

$$\|\nu\|_{D_d} \lesssim d^{\frac{1}{2}-5\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + d^{-1}\|v\|_{D_d} + cd^{-\frac{1}{2}-3\epsilon}\|v\|_{D_d}^{\frac{3}{2}-3\epsilon} + cd^{-2\epsilon}\|v\|_{D_d}^{2-2\epsilon}, \tag{3.4.14}$$

$$\begin{aligned}[v+\Upsilon]_{1-2\epsilon,D_d,d} \lesssim\ & d^{\frac{1}{2}-3\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + c\|v\|_{D_d}^{2-2\epsilon} + d^{-1+2\epsilon}\|v\|_{D_d} \\ &+ cd^{-\frac{1}{2}-\epsilon}\|v\|_{D_d}^{\frac{3}{2}-3\epsilon},\end{aligned} \tag{3.4.15}$$

$$[v]_{\frac{1}{2}-3\epsilon,D_d,d} \lesssim d^{1-2\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} + d^{-\frac{1}{2}+3\epsilon}\|v\|_{D_d} + c\|v\|_{D_d}^{\frac{3}{2}-3\epsilon} \tag{3.4.16}$$

$$+ cd^{\frac{1}{2}+\epsilon}\|v\|_{D_d}^{2-2\epsilon},$$

$$[\nu]_{\frac{1}{2}-5\epsilon,D_d} \lesssim [U]_{\frac{3}{2}-5\epsilon,D_d}(1 + cd^{1-2\epsilon}\|v\|_D^{1-2\epsilon}) + c^2(\|v\|_{D_d}^{\frac{5}{2}-5\epsilon} + d^{\frac{1}{2}+\epsilon}\|v\|_{D_d}^{3-4\epsilon})$$
$$+ d^{-\frac{3}{2}+5\epsilon}\|v\|_{D_d} + cd^{-1+2\epsilon}\|v\|_{D_d}^{\frac{3}{2}-3\epsilon} + cd^{-\frac{1}{2}+3\epsilon}\|v\|_{D_d}^{2-2\epsilon}. \quad (3.4.17)$$

We inject those in the right-hand side of (3.4.8) and we bound positive powers of d by powers of d_0. A few computations permit to reduce the result to the following expression:

$$\sup_{d\leqslant d_0} d^{\frac{3}{2}-5\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} \lesssim c\sum_h d_0^h\|v\|_D^{h+1} + c\sup_{d\leqslant d_0} d^{\frac{3}{2}-5\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d}\sum_l d_0^l\|v\|_D^l, \quad (3.4.18)$$

where the index of the sum h is taken in a finite subset of $[0, \frac{9}{2} - 11\epsilon]$ and the index l in $\{1-2\epsilon, \frac{2}{3}-\epsilon, 2-4\epsilon\}$. If we set

$$d_0 = \|v\|_D^{-1}, \quad (3.4.19)$$

we see that we can get rid of $\sup_{d\leqslant d_0} d^{\frac{3}{2}-5\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d}$ in the right-hand side under a first smallness condition on c depending on the constant implicit in $\lesssim$. Note that if (3.4.19) defines d_0 too large, it means that we already have a bound on $\|v\|_D$. If the smallness condition on c is satisfied, we have:

$$\sup_{d\leqslant d_0} d^{\frac{3}{2}-5\epsilon}[U]_{\frac{3}{2}-5\epsilon,D_d} \lesssim c\|v\|_D. \quad (3.4.20)$$

In this equation and in the following, $\lesssim$ does not depend on this first condition on c. Applying this to Equations (3.4.14) to (3.4.17) gives

$$\sup_{d\leqslant d_0} d\|\nu\|_{D_d} \lesssim c\|v\|_D, \quad (3.4.21)$$

$$\sup_{d\leqslant d_0} d^{1-2\epsilon}[v + \Upsilon]_{1-2\epsilon,D_d,d} \lesssim c\|v\|_D, \quad (3.4.22)$$

$$\sup_{d\leqslant d_0} d^{\frac{1}{2}-3\epsilon}[v]_{\frac{1}{2}-3\epsilon,D_d,d} \lesssim c\|v\|_D, \quad (3.4.23)$$

$$\sup_{d\leqslant d_0} d^{\frac{3}{2}-5\epsilon}[\nu]_{\frac{1}{2}-5\epsilon,D_d} \lesssim c\|v\|_D. \quad (3.4.24)$$

Applying estimates (3.4.20) and (3.4.21)-(3.4.24) to (3.4.2) and (3.4.3), we have, for $d \leqslant d_0$ and $B(x,T) \in D_d$,

$$|(v^2\Upsilon)_T(x)| \lesssim c^2(d^{-1+2\epsilon}T^{\frac{1}{2}-3\epsilon}\|v\|_D^{\frac{5}{2}-\epsilon} + d^{-\frac{1}{2}+3\epsilon}T^{\frac{1}{2}-7\epsilon}\|v\|_D^{3-4\epsilon}) \quad (3.4.25)$$
$$+ c(T^{-\frac{1}{2}-\epsilon}\|v\|_D^{\frac{5}{2}-\epsilon} + T^{-4\epsilon}\|v\|_D^{3-4\epsilon} + T^{\frac{1}{2}-7\epsilon}\|v\|_D^{\frac{7}{2}-7\epsilon}).$$

and

$$
\begin{aligned}
&|(v\mathbb{V})_T(x) + 3C_2(v_T + \mathbb{I}_T)(x)| \lesssim c\|v\|_D^{2-2\epsilon} T^{-1-2\epsilon} \\
&\quad + c^2 T^{\frac{1}{2}-7\epsilon}\Big(d^{-\frac{1}{2}+3\epsilon}\|v\|_D^{3-4\epsilon} + d^{-\frac{3}{2}+5\epsilon}\|v\|_D^{2-2\epsilon}\Big) \\
&\quad + c(\|v\|_D^{\frac{5}{2}-5\epsilon} T^{-\frac{1}{2}-5\epsilon} + T^{-4\epsilon}\|v\|_D^{3-4\epsilon}) + c^2 d^{-1}\|v\|_D^{2-2\epsilon} T^{-2\epsilon}. \qquad (3.4.26)
\end{aligned}
$$

In this last estimate, we have used triangle inequality to get $\|v\mathbb{V}_T\|_{D_{d+T}}$ out of the left hand side, and then used the assumption (3.4.1) to bound it.

3.4.4 Application of Lemma 3.2.7

We now go back to the original equation, and start to study large scale behaviour. We convolve the equation (3.2.18) with Ψ_L:

$$
\begin{aligned}
(\partial_t - \Delta)(v)_L = &-(v_L)^3 - 3(v^2\mathbb{I})_L - 3(v\mathbb{V})_L - 9C_2(v_L + \mathbb{I}_L) - (\mathbb{V})_L \qquad (3.4.27) \\
&+ ((v_L)^3 - (v^3)_L).
\end{aligned}
$$

Lemma 3.2.7 implies that for all $r \geqslant 0$ and $0 < R' < R$ such that $r + R < \frac{1}{2}$, we have

$$
\begin{aligned}
\|(v)_L\|_{P_{r+R}} \lesssim \max\Big\{ &\frac{1}{R-R'}, \|(v_L)^3 - (v^3)_L\|_{P_{r+R'}}^{\frac{1}{3}}, \|(v^2\mathbb{I})_L\|_{P_{r+R'}}^{\frac{1}{3}}, \\
&\|(v\mathbb{V})_L + 3C_2(v_L + \mathbb{I}_L)\|_{P_{r+R'}}^{\frac{1}{3}}, \|(\mathbb{V})_L\|_{P_{r+R'}}^{\frac{1}{3}} \Big\}. \qquad (3.4.28)
\end{aligned}
$$

The goal is now to balance the commutator and the renormalized powers of the noise term by choosing the parameter L appropriately. We first mention that applying (3.2.9) and then Assumption 3.4.1 gives the bound:

$$
\begin{aligned}
\|v\|_{P_{r+R}} \lesssim \max\Big\{ &\frac{1}{R-R'}, L^{\frac{1}{2}-3\epsilon}[v]_{\frac{1}{2}-3\epsilon, P_{r+R-L}, 2L}, \|(v_L)^3 - (v^3)_L\|_{P_{r+R'}}^{\frac{1}{3}}, \\
&\|(v^2\mathbb{I})_L\|_{P_{r+R'}}^{\frac{1}{3}}, \|(v\mathbb{V})_L + 3C_2(v_L + \mathbb{I}_L)\|_{P_{r+R'}}^{\frac{1}{3}}, \|(\mathbb{V})_L\|_{P_{r+R'}}^{\frac{1}{3}} \Big\}. \qquad (3.4.29)
\end{aligned}
$$

We need estimates on the commutator $(v_L)^3 - (v^3)_L$. This is easily obtained as v is $C^{\frac{1}{2}-3\epsilon}$, using the moment bounds (3.2.8) and (3.2.9). For any $z \in P_{r+R'}$,

$$
\begin{aligned}
&((v_L)^3 - (v^3)_L)(z) = \int \Psi_L(z-\overline{z}) \left(v_L(z)^3 - v(\overline{z})^3\right) d\overline{z} \\
&= \int \Psi_L(z-\overline{z}) \int_0^1 (v_L(z) - v(\overline{z}))\, 3\left(\lambda v_L(z) + (1-\lambda)v(\overline{z})\right)^2 d\lambda d\overline{z}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 3\|v\|^2_{B(z,L)} \int \Psi_L(z-\overline{z})\,(v_L(z)-v(z)+v(z)-v(\overline{z}))\,d\overline{z}\\
&\leqslant 3\|v\|^2_{B(z,L)} \int \Psi_L(z-\overline{z})\left(L^{\frac{1}{2}-3\epsilon}+d(z,\overline{z})^{\frac{1}{2}-3\epsilon}\right)[v]_{\frac{1}{2}-3\epsilon,B(z,L)}d\overline{z}\\
&\leqslant 6\|v\|^2_{B(z,L)} L^{\frac{1}{2}-3\epsilon}[v]_{\frac{1}{2}-3\epsilon,B(z,L)}.
\end{aligned}
$$

Since this is true for all $z \in P_{r+R'}$,

$$
\|(v_L)^3-(v^3)_L\|_{P_{r+R'}} \lesssim \|v\|^2_{P_{r+R'-L}} L^{\frac{1}{2}-3\epsilon}[v]_{\frac{1}{2}-3\epsilon,P_{r+R'-L},2L}. \tag{3.4.30}
$$

In conclusion of this step,

$$
\begin{aligned}
&\|v\|_{P_{r+R}} \lesssim\\
&\quad \max\Big\{\frac{1}{R-R'}, L^{\frac{1}{2}-3\epsilon}[v]_{\frac{1}{2}-3\epsilon,P_{r+R-L},2L}, L^{\frac{1}{6}-\epsilon}\|v\|^{\frac{2}{3}}_{P_{r+R'-L}}[v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon,P_{r+R'-L},2L},\\
&\qquad \|(v^2\mathbb{1})_L\|^{\frac{1}{3}}_{P_{r+R'}}, \|(v\mathbb{V})_L+3C_2(v_L+\mathbb{1}_L)\|^{\frac{1}{3}}_{P_{r+R'}}, \|(\mathbb{V})_L\|^{\frac{1}{3}}_{P_{r+R'}}\Big\}.
\end{aligned} \tag{3.4.31}
$$

3.4.5 Choice of scale

We now apply the assumption (3.4.1) with $\tau = \mathbb{V}$ and the results from the previous steps, (3.4.23),(3.4.25) and (3.4.26) to equation (3.4.31). In (3.4.31) we choose

$$
R' = d_0 \text{ and } L = \frac{d_0}{k},
$$

for some $k > 2$ to be specified. Recall that $d_0 = \frac{1}{\|v\|_{P_r}}$, as set in (3.4.19). In the left-hand side of (3.4.23),(3.4.25) and (3.4.26) we make the particular choice

$$
d = d_0\frac{k-1}{k}.
$$

Since $k > 2$ we have $d \sim d_0$, and we also have $L + d = d_0$ so $\|v\|_{P_{r+R'-L}} = \|v\|_{D_d}$. Equation (3.4.31) simplifies to

$$
\begin{aligned}
\|v\|_{P_{r+R}} \leqslant C\max\Big\{&\frac{1}{R-R'}, \|v\|_{P_r}k^{-\frac{1}{2}+3\epsilon}, \|v\|_{P_r}k^{-\frac{1}{6}+\epsilon},\\
&\|v\|_{P_r}\Big[c^2\Big(k^{-\frac{1}{2}+3\epsilon}+k^{-1+6\epsilon}+k^{-\frac{1}{2}+7\epsilon}+k^{-\frac{1}{2}+7\epsilon}+k^{2\epsilon}\Big)\\
&+c\Big(k^{-\frac{1}{2}+\epsilon}+k^{4\epsilon}+k^{-\frac{1}{2}+7\epsilon}+k^{1+\epsilon}+k^{\frac{1}{2}+5\epsilon}+k^{\frac{3}{2}+3\epsilon}\Big)\Big]^{\frac{1}{3}}\Big\},
\end{aligned} \tag{3.4.32}
$$

for some constant $C > 1$. We see that we can choose k large and then impose another smallness condition on c to get,

$$\|v\|_{P_{R+r}} \leqslant \max\Big\{\frac{C}{R-R'}, \frac{1}{2}\|v\|_{P_r}\Big\}. \tag{3.4.33}$$

3.4.6 Iterating the result

If we have $R \geqslant 2R'$, then we can rewrite equation (3.4.33) for $r = 0$ as

$$\|v\|_{P_R} \leqslant \max\Big\{\frac{2C}{R}, \frac{\|v\|_P}{2}\Big\}. \tag{3.4.34}$$

The first argument of the maximum (3.4.34) is equal to the second one for

$$R = R_1 := \frac{4C}{\|v\|_P}.$$

This is not in contradiction with $R \geqslant 2R' = \frac{2}{\|v\|_P}$ as $C > 1$. We now define a finite set $0 = R_0 < \ldots < R_N = \frac{1}{2}$ by setting

$$R_{n+1} - R_n = 4C\|v\|^{-1}_{P_{R_n}},$$

as long as the times R_{n+1} defined this way stay strictly less than $\frac{1}{2}$. We terminate the sequence once this algorithm would produce a $R_{n+1} \geqslant \frac{1}{2}$ in which case we set $R_{n+1} = R_N = \frac{1}{2}$ or once the Assumption (3.4.1) does not hold for $D = R_n$. Note that $4C\|v\|^{-1}_{P_{R_n}}$ is increasing in n so the sequence necessarily terminates after finitely many steps. Equation (3.4.33) applied with $r = R_{n-1}$ for $n = 1...N$ then gives the bounds for smaller and smaller parabolic boxes.

$$\|v\|_{P_{R+R_{n-1}}} \leqslant \max\Big\{\frac{2C}{R}, \frac{1}{2}\|v\|_{P_{R_{n-1}}}\Big\}, \tag{3.4.35}$$

hence for $R = R_n - R_{n-1}$,

$$\|v\|_{P_{R_n}} \leqslant \frac{1}{2}\|v\|_{P_{R_{n-1}}}. \tag{3.4.36}$$

We now show that the bound (3.2.19) in Theorem 3.2.1 holds for all $R = R_n, n \in \{0, ..., N\}$. If Assumption (3.4.1) does not hold for $D = R_N$ it is immediate, in the other case for $k \leqslant n$, $\|v\|_{P_{R_n}} \leqslant \|v\|_{P_{R_k}} 2^{k-n}$ and hence

$$R_n = \sum_{k=0}^{n-1} R_{k+1} - R_k = \sum_{k=0}^{n-1} 4C\|v\|^{-1}_{P_{R_k}} \leqslant 4C\|v\|^{-1}_{P_{R_n}} \sum_{k=0}^{n-1} 2^{k-n} \lesssim \|v\|^{-1}_{P_{R_n}}. \tag{3.4.37}$$

For the end point R_N we have either $R_{N-1} \geqslant \frac{1}{4}$ or $R_N - R_{N-1} \geqslant \frac{1}{4}$. In the first case we invoke (3.4.37) for $n = N-1$ and in the second case the definition of $R_{n+1} - R_n$, in both cases yielding a bound on $\|v\|_{P_{R_{N-1}}}$. Finally for values $R \in (R_n, R_{n+1})$, we use the definition of $R_{n+1} - R_n$:

$$R \leqslant R_{n+1} = R_{n+1} - R_n + R_n \lesssim \|v\|_{P_{R_n}}^{-1} + R_n \lesssim \|v\|_{P_R}^{-1}.$$

This concludes the proof of the theorem.

3.5 Proof of lemmas

3.5.1 Reconstruction

Proof of Theorem 3.2.8

For this proof, we will use the following notations for f a function of one variable and F a function of two variables:

$$[F, (\cdot)_T](x) = \int \Psi_T(x-y)F(y,x)dy \tag{3.5.1}$$

This is the only place where our particular choice of convolution kernel is crucial. It enables us to use the following factorisation:

$$\begin{aligned}
&\Big|[F, (\cdot)_{T2^{-n}}](x_1) - \Big([F, (\cdot)_{T2^{-n-1}}]\Big)_{T2^{-n},1}(x_1)\Big| \\
&\quad = \Big|\int\int \Psi_{T2^{-n-1}}(x_2-y)\Phi_{T2^{-n-1}}(x_1-x_2)(F(y,x_1)-F(y,x_2))dydx_2\Big| \\
&\quad \leqslant \sum_{\beta\in A} C_\beta \int \Phi_{T2^{-n-1}}(x_1-x_2)d(x_1,x_2)^{\gamma_\beta-\beta}(T2^{-n-1})^\beta dx_2 \\
&\quad \leqslant \sum_{\beta\in A} C_\beta (T2^{-n-1})^{\gamma_\beta}.
\end{aligned}$$

This proves the convergence of $[F, (\cdot)_{T2^{-n}}]$ to $f : y \mapsto F(y,y)$ and justifies the bound on the following telescopic sum, obtained once more thanks to the semi-group property of our kernel:

$$\begin{aligned}
\Big|[F, (\cdot)_T] - [F, (\cdot)_{T2^{-N}}]_{T,N-1}\Big| &= \Big|\sum_{n=0}^{N}\Big([F, (\cdot)_{T2^{-n}}] - [F, (\cdot)_{T2^{-n-1}}]_{T2^{-n},1}\Big)_{T,n}\Big| \\
&\leqslant \sum_{n=0}^{N}\sum_{\beta\in A} C_\beta (T2^{-n-1})^{\gamma_\beta} \lesssim \sum_{\beta\in A} C_\beta T^{\gamma_\beta},
\end{aligned}$$

where the constant in "$\lesssim$" depends only on γ (in particular not on N), thus proving the theorem.

Proof of Lemma 3.2.9

To obtain a bound on $(v^2\mathbf{\dagger})_T(y)$ we implement the following expansion.

$$\begin{aligned}(v^2\mathbf{\dagger})_T(x) = (v^2\mathbf{\dagger})_T(x) - v(x)^2\mathbf{\dagger}_T(x) - 2v(x)((\mathbf{\Psi}(x) - \mathbf{\Psi})\mathbf{\dagger})_T(x)\\ + v(x)^2\mathbf{\dagger}_T(x) + 2v(x)((\mathbf{\Psi}(x) - \mathbf{\Psi})\mathbf{\dagger})_T(x).\end{aligned}$$

From the bound (3.2.16) we have

$$|v(x)^2\mathbf{\dagger}_T(x)| \leqslant \|v\|^2_{B(x,T)}[\mathbf{\dagger}]_{-\frac{1}{2}-\epsilon}T^{-\frac{1}{2}-\epsilon} \tag{3.5.2}$$

$$|v(x)((\mathbf{\Psi}(x) - \mathbf{\Psi})\mathbf{\dagger})_T(x)| \leqslant \|v\|_{B(x,T)}[\mathbf{\Psi}]_{-4\epsilon}T^{-4\epsilon}. \tag{3.5.3}$$

To bound the remaining part, we will apply Theorem 3.2.8 and to that end we set

$$F(y, x_1) = v(x_1)^2\mathbf{\dagger}(y) + 2v(x_1)(\mathbf{\Psi}(x_1) - \mathbf{\Psi}(y))\mathbf{\dagger}(y). \tag{3.5.4}$$

Then

$$\begin{aligned}F(y, x_1) - F(y, x_2) =&(v(x_1) + v(x_2))(v(x_1) - v(x_2) + \mathbf{\Psi}(x_1) - \mathbf{\Psi}(x_2))\mathbf{\dagger}(y)\\ &+ (v(x_1) - v(x_2))(\mathbf{\Psi}(x_1) - \mathbf{\Psi}(x_2))\mathbf{\dagger}(y)\\ &+ 2(v(x_1) - v(x_2))(\mathbf{\Psi}(x_1) - \mathbf{\Psi}(y))\mathbf{\dagger}(y).\end{aligned}$$

By definition of $\mathbf{\Psi}$ (3.2.16) this gives, for $x_1, x_2 \in B(x, T - t)$

$$\begin{aligned}|\int \Psi_t(x_1 - y)(F(y, x_1) - F(y, x_2))dy|&\\ \leqslant&2\|v\|_{D_d}[v + \mathbf{\Psi}]_{1-2\epsilon,B(x,T)}d(x_1, x_2)^{1-2\epsilon}[\mathbf{\dagger}]_{-\frac{1}{2}-\epsilon}t^{-\frac{1}{2}-\epsilon}\\ &+ [v]_{\frac{1}{2}-3\epsilon,B(x,T)}[\mathbf{\Psi}]_{\frac{1}{2}-3\epsilon}d(x_1, x_2)^{1-6\epsilon}[\mathbf{\dagger}]_{-\frac{1}{2}-\epsilon}t^{-\frac{1}{2}-\epsilon}\\ &+ [v]_{\frac{1}{2}-3\epsilon,B(x,T)}d(x_1, x_2)^{\frac{1}{2}-3\epsilon}[\mathbf{\Psi}]_{-4\epsilon}t^{-4\epsilon}.\end{aligned}$$

Hence by Theorem 3.2.8, we have the bound

$$\begin{aligned}|(v^2\mathbf{\dagger})_T(x) - v(x)^2\mathbf{\dagger}_T(x) + 2v(x)((\mathbf{\Psi}(x) - \mathbf{\Psi})\mathbf{\dagger})_T(x)|&\\ \lesssim T^{\frac{1}{2}-3\epsilon}2\|v\|_{D_d}[v + \mathbf{\Psi}]_{1-2\epsilon,B(x,T)}[\mathbf{\dagger}]_{-\frac{1}{2}-\epsilon}&\\ +T^{\frac{1}{2}-7\epsilon}[v]_{\frac{1}{2}-3\epsilon,B(x,T)}([\mathbf{\Psi}]_{\frac{1}{2}-3\epsilon}[\mathbf{\dagger}]_{-\frac{1}{2}-\epsilon} + [\mathbf{\Psi}]_{-4\epsilon}).&\end{aligned} \tag{3.5.5}$$

Together with bounds (3.5.2) to (3.5.3), we get Lemma 3.2.9.

Proof of Lemma 3.2.10

The last quantity we need to bound is:

$$\begin{aligned}((v-v(x))\mathbb{V})_T(x) + 3C_2(v_T(x) + \mathbb{1}_T(x)) = \\ [\overline{U}\mathbb{V},(\cdot)_T](x) - 3C_2(v-v_T)(x) - ((\mathbb{\Upsilon}-\mathbb{\Upsilon}(x))\mathbb{V} - 3C_2\mathbb{1})_T(x) \\ - 3v(x)((\mathbb{\Upsilon}-\mathbb{\Upsilon}(x))\mathbb{V} - 3C_2)_T(x) + \nu(x).(X(\cdot-x)\mathbb{V})_T(x),\end{aligned}$$

where $\overline{U}(y,x) = U(y,x) - \nu(x).X(y-x)$ and ν is optimal in the definition of $[U]_{1+3\epsilon,D}$. From the bounds (3.2.17), (3.2.15) and (3.2.14) we have

$$|((\mathbb{\Upsilon}-\mathbb{\Upsilon}(x))\mathbb{V} - 3C_2\mathbb{1})_T(x)| \leqslant [\mathbb{\Psi}]_{-\frac{1}{2}-5\epsilon}T^{-\frac{1}{2}-5\epsilon}, \tag{3.5.6}$$

$$|v(x)((\mathbb{\Upsilon}-\mathbb{\Upsilon}(x))\mathbb{V} - 3C_2)_T(y) \leqslant \|v\|_{B(x,T)}[\mathbb{\Psi}]_{-4\epsilon}T^{-4\epsilon}, \tag{3.5.7}$$

$$|\nu(x).(X(\cdot-y)\mathbb{V})_T(x)| \leqslant \|\nu\|_{B(x,T)}[\mathbb{V}_x]_{-2\epsilon}T^{-2\epsilon}. \tag{3.5.8}$$

To bound the remaining part, we will apply Theorem 3.2.8 and to that end we set

$$\begin{aligned}F(y,x_1) = \Big(v(x_1) + \mathbb{\Upsilon}(x_1) - \mathbb{\Upsilon}(y) + 3v(x_1)(\mathbb{\Upsilon}(x_1) - \mathbb{\Upsilon}(y)) \\ -\nu(x_1).X(x_1-y)\Big)\mathbb{V}(y) - 3C_2(v(x_1)-v(y)).\end{aligned} \tag{3.5.9}$$

Then for $x_1, x_2 \in B(x, T-t)$,

$$\begin{aligned}F(y,x_1) - F(y,x_2) =& (3(v(x_1)-v(x_2))((\mathbb{\Upsilon}(y)-\mathbb{\Upsilon}(x_2))\mathbb{V}(y) - C_2) \\ &+ U(x_1,x_2)\mathbb{V}(y) - (\nu(x_1)-\nu(x_2)).X(y-x_2)\mathbb{V}(y).\end{aligned}$$

By definition of $\mathbb{\Psi}$ (3.2.15) and $\mathbb{V}_x$ (3.2.14), this gives

$$\begin{aligned}\int \Psi_t(x_1-y)(F(y,x_1)-F(y,x_2))dy \\ \leqslant& 3[v]_{\frac{1}{2}-3\epsilon,B(x,T)}d(x_1,x_2)^{\frac{1}{2}-3\epsilon}[\mathbb{\Psi}]_{-4\epsilon}t^{-4\epsilon} \\ &+ [U]_{\frac{3}{2}-5\epsilon,B(x,T)}d(x_1,x_2)^{\frac{3}{2}-5\epsilon}[\mathbb{V}]_{-1-2\epsilon}t^{-1-2\epsilon} \\ &+ [\nu]_{\frac{1}{2}-5\epsilon,B(x,T)}d(x_1,x_2)^{\frac{1}{2}-5\epsilon}[\mathbb{V}_x]_{-2\epsilon}t^{-2\epsilon}.\end{aligned}$$

Hence by Theorem 3.2.8, we have the bound $v\mathbb{V}$ such that

$$\begin{aligned}\|[\overline{U}\mathbb{V},(\cdot)_T](x) - 3C_2(v-v_T)(x)\| \lesssim T^{\frac{1}{2}-7\epsilon}\Big(3[v]_{\frac{1}{2}-3\epsilon,B(x,T)}[\mathbb{\Psi}]_{-4\epsilon} \\ + [U]_{\frac{3}{2}-5\epsilon,B(x,T)}[\mathbb{V}]_{-1-2\epsilon} + [\nu]_{\frac{1}{2}-5\epsilon,B(x,T)}[\mathbb{V}_x]_{-2\epsilon}\Big).\end{aligned} \tag{3.5.10}$$

Together with bounds (3.5.6) to (3.5.8), we get the lemma 3.2.10.

multiplication by C^2 function

This is a useful lemma that shows a simple application of the reconstruction lemma.

Lemma 3.5.1. *Let g be a C^2 function and let h be a distribution of regularity $\gamma > -2$. Then $gh \in C^\gamma$ with $[gh]_\gamma \lesssim [h]_\gamma(1 + \|g''\|)$*

Proof. We do the proof in one dimension for simplicity. It is identical in higher dimension. Once again, we do not concern ourselves with existence so we may assume that g and h are actually smooth functions, but we only allow a control with their given regularity.

Take the Taylor expansion of g: $g(y) = g(x) + g'(y)(y-x) + \mathrm{Err}(x,y)$ where $|\mathrm{Err}(x,y)| \leqslant \|g''\||x-y|^2$. Then define $F(y,x) = (g(x) + g'(y)(y-x))h(y)$. We have by triangle inequalities

$$\begin{aligned}|F(y,x_1) - F(y,x_2)| &= |h(y)(g(x_1) - g(x_2) + g'(y)(x_2 - x_1))| \\ &\leqslant |h(y)|\|g''\|(|x_1 - x_2| + |x_2 - y|)^2\end{aligned}$$

Therefore

$$\Big|\int \Psi_l(x_2 - y)(F(y,x_1) - F(y,x_2))dy\Big| \leqslant 4\|g''\|[h]_\gamma(|x_1 - x_2|^2 l^\gamma + l^{\gamma+2},$$

and the reconstruction lemma applies, giving

$$\Big|\int \Psi_L(x-y)(g(x) - g(y) + g'(y)(y-x))h(y)dy\Big| \lesssim \|g''\|[h]_\gamma L^{\gamma+2},$$

which in particular guarantees that $gh \in C^\gamma$ with $[gh]_\gamma \lesssim [h]_\gamma(1 + \|g''\|)$. □

3.5.2 Schauder theory

We only prove the most complex of the lemmas, the proof of the others an be understood by removing the unnecessary arguments from this one.

Proof of Lemma 3.2.3

STEP 1. We claim that for all base points x and scales T, R and L with $R \leqslant \frac{L}{2}$ and such that $B(x,L) \subset D$, it holds:

$$\inf_l \|U_T(\cdot,x) - l\|_{B(x,R)} \tag{3.5.11}$$

$$\lesssim \frac{R^2}{L^2} \inf_l \|U_T(\cdot,x) - l\|_{B(x,L)} + L^2 M^{(1)}_{\{x\},L} \sum_{\beta\in A} T^{\beta-2} L^{\kappa-\beta},$$

where the infimum runs over all affine functions $l(y) = C.X(y-x) + c$. To prove this, we define a decomposition $U_T(\cdot,x) = u_> + u_<$ where $u_>$ is the solution to

$$(\partial_t - \Delta) u_> = \mathbf{1}_{B(x,L)} (\partial_t - \Delta) U_T(\cdot,x).$$

with Dirichlet boundary conditions. By standard estimates for the heat equation [49, Cor.8.1.5],

$$\|u_>\|_{B(x,L)} \lesssim L^2 \|(\partial_t - \Delta) U_T(\cdot,x)\|_{B(x,L)} \overset{(3.2.37)}{\leqslant} L^2 M^{(1)}_{\{x\},L} \sum_{\beta\in A} T^{\beta-2} L^{\kappa-\beta}. \tag{3.5.12}$$

As $(\partial_t - \Delta) u_< = 0$ on $B(x,L)$ for $\partial \in \{\partial_t, \partial_i\partial_j\}$ a differential operator of order 1 in time or 2 in space,

$$\|\partial u_<\|_{B(x,R)} \leqslant L^{-2} \|u_< - l_>\|_{B(x,L)},$$

for any affine function $l_>$, where we used $R \leqslant \frac{L}{2}$, and the fact that the differential operators used cancel the spatial linear functional. Next we define a concrete affine function $l_<$ via $l_<(y) := u_<(x) + \triangledown u_<(x).X(y-x)$ and observe, using Taylor's formula,

$$\begin{aligned}
\|u_< - l_<\|_{B(x,R)} \leqslant & R^2 \|Du_<\|_{B(x,R)} \\
\leqslant & \frac{R^2}{L^2} \|u_< - l_>\|_{B(x,L)} \\
\leqslant & \frac{R^2}{L^2} \|U_T(\cdot,x) - l_>\|_{B(x,L)} + \|u_>\|_{B(x,L)}.
\end{aligned}$$

Using the triangle inequality once more and (3.5.12) gives:

$$\begin{aligned}
\|U_T(\cdot,x) - l_>\|_{B(x,R)} \leqslant & \|u_< - l_<\|_{B(x,R)} + \|u_>\|_{B(x,R)} \\
\lesssim & \frac{R^2}{L^2} \|U_T(\cdot,x) - l_>\|_{B(x,L)} + \|u_>\|_{B(x,L)} \\
\overset{(3.5.12)}{\leqslant} & \frac{R^2}{L^2} \|U_T(\cdot,x) - l_>\|_{B(x,L)} + L^2 M^{(1)}_{\{x\},L} \sum_{\beta\in A} T^{\beta-2} L^{\kappa-\beta},
\end{aligned}$$

which implies (3.5.13)

STEP 2. We claim that for all base points x and scales T, L, it holds:

$$\|U_T(\cdot,x)-U(\cdot,x)\|_{B(x,R)} \leqslant M^{(2)}_{\{x\},R,T}\sum_{\beta\in A} R^\beta T^{\kappa-\beta} + T^\kappa [U]_{\kappa,B(x,R),T}. \quad (3.5.13)$$

Indeed, since Ψ is symmetric, it integrates to 0 against linear functions hence for any $y \in B(x,R)$, we have

$$\begin{aligned}
|U_T(y,x)-U(y,x)| &= \Big|\int \Psi_T(y-z)(U(z,x)-U(y,x))dz\Big| \\
&= \inf_{\nu(y)}\Big|\int \Psi_T(y-z)(U(z,x)-U(y,x)-U(z,y)+\lambda(y,x)\cdot X(z-y))dz \\
&\quad + \int \Psi_T(y-z)(U(z,y)-\nu(y).X(z-y))dz\Big| \\
&\leqslant M^{(2)}_{\{x\},R,T}\sum_{\beta\in A} d(y,x)^\beta \int \Psi_T(y-z)d(z,y)^{\kappa-\beta}dz \\
&+\Big(\sup_{y\in B(x,R)}\inf_{\nu(y)}\sup_{z\in B(y,T)} d(y,z)^{-\kappa}|U(z,y)-\nu(y).X(z-y)|\Big) \\
&\quad \times \int \Psi_T(y-z)d(z,y)^\kappa dz.
\end{aligned}$$

STEP 3. We prove

$$\begin{aligned}
&\sup_{R\leqslant \frac{\epsilon d}{2}} R^{-\kappa}\inf_l \|U(\cdot,x)-l\|_{B(x,R)} \\
&\lesssim \sum_{\beta\in A}\Big(M^{(1)}_{\{x\},\frac{d}{2}}\epsilon^{-4+2\beta-\kappa} + M^{(2)}_{\{x\},\frac{\epsilon d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{\kappa-\beta} + M^{(2)}_{\{x\},\frac{d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{2(\kappa-\beta)}\Big) \\
&\qquad + \epsilon^{2-2\kappa}\frac{d^{-\kappa}}{2-\kappa}\|U(\cdot,x)\|_{B(x,\frac{d}{2}(1+\epsilon^2))} + (\epsilon^\kappa+\epsilon^{2+\kappa})[U]_{\kappa,B(x,\frac{\epsilon d}{2}),\frac{\epsilon^2 d}{2}}. \quad (3.5.14)
\end{aligned}$$

Multiplying Equation (3.5.11) by $R^{-\kappa}$ and fixing the length ratios $R=\epsilon L=\epsilon^{-1}T$ for some $\epsilon \leqslant \frac{1}{2}$ to be fixed below, we get for any point $x \in D_d$ and length $L \leqslant \frac{d}{2}$,

$$\begin{aligned}
&R^{-\kappa}\inf_l \|U_T(\cdot,x)-l\|_{B(x,R)} \\
&\lesssim \epsilon^{2-\kappa}L^{-\kappa}\inf_l \|U_T(\cdot,x)-l\|_{B(x,L)} + \sum_{\beta\in A} M^{(1)}_{D_d,L}\epsilon^{-4+2\beta-\kappa}.
\end{aligned}$$

Taking the supremum over $L \leqslant \frac{d}{2}$ while keeping the ratios $R=\epsilon L=\epsilon^{-1}T$ fixed we get

$$\sup_{R\leqslant\frac{\epsilon d}{2}} R^{-\kappa}\inf_l \|U_T(\cdot,x)-l\|_{B(x,R)}$$

$$
\begin{aligned}
&\lesssim \epsilon^{2-\kappa} \sup_{L\leqslant \frac{d}{2}} L^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,L)} + \sup_{L\leqslant \frac{d}{2}} \sum_{\beta\in A} M^{(1)}_{D_d,L}\epsilon^{-4+2\beta-\kappa} \\
&\leqslant \epsilon^{2-\kappa} \sup_{L\leqslant \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,L)} + \sup_{L\leqslant \frac{d}{2}} \sum_{\beta\in A} M^{(1)}_{D_d,L}\epsilon^{-4+2\beta-\kappa} \\
&+ \epsilon^{2-\kappa} \sup_{\frac{\epsilon d}{2}\leqslant L\leqslant \frac{d}{2}} L^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,L)}.
\end{aligned}
$$

The last term is bounded by

$$
\epsilon^{2-\kappa}\Big(\frac{\epsilon d}{2}\Big)^{-\kappa}\|U_T(\cdot,x)\|_{B(x,\frac{d}{2})} \leqslant \epsilon^{2-2\kappa}\frac{d^{-\kappa}}{2^{-\kappa}}\|U(\cdot,x)\|_{B(x,\frac{d}{2}(1+\epsilon^2))}.
$$

Hence we have

$$
\begin{aligned}
&\sup_{R\leqslant \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,R)} \\
&\qquad \lesssim \epsilon^{2-\kappa} \sup_{L\leqslant \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,L)} \\
&\qquad\quad + \sum_{\beta\in A} M^{(1)}_{D_d,\frac{d}{2}}\epsilon^{-4+2\beta-\kappa} + \epsilon^{2-2\kappa}\frac{d^{-\kappa}}{2^{-\kappa}}\|U(\cdot,x)\|_{B(x,\frac{d}{2}(1+\epsilon^2))},
\end{aligned}
$$

where the ratios between L and T, and R and T are fixed only within the supremum operators. Applying Equation (3.5.13) gives

$$
\begin{aligned}
&\sup_{R\leqslant \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U(\cdot,x)-l\|_{B(x,R)} \\
&\leqslant \sup_{R\leqslant \frac{\epsilon d}{2}} R^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,R)} \\
&\qquad + M^{(2)}_{\{x\},\frac{\epsilon d}{2},\frac{\epsilon^2 d}{2}} \sum_{\beta\in A}\epsilon^{\kappa-\beta} + \epsilon^{\kappa}[U]_{\kappa,B(x,\frac{\epsilon d}{2}),\frac{\epsilon^2 d}{2}} \\
&\lesssim \sum_{\beta\in A}\Big(M^{(1)}_{\{x\},\frac{d}{2}}\epsilon^{-4+2\beta-\kappa} + M^{(2)}_{D_d,\frac{\epsilon d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{\kappa-\beta}\Big) \\
&\qquad + \epsilon^{2-2\kappa}\frac{d^{-\kappa}}{2^{-\kappa}}\|U(\cdot,x)\|_{B(x,\frac{d}{2}(1+\epsilon^2))} + \epsilon^{\kappa}[U]_{\kappa,B(x,\frac{\epsilon d}{2}),\frac{\epsilon^2 d}{2}} \\
&\qquad + \epsilon^{2-\kappa} \sup_{L\leqslant \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U_T(\cdot,x)-l\|_{B(x,L)} \\
&\lesssim \sum_{\beta\in A}\Big(M^{(1)}_{\{x\},\frac{d}{2}}\epsilon^{-4+2\beta-\kappa} + M^{(2)}_{\{x\},\frac{\epsilon d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{\kappa-\beta} + M^{(2)}_{\{x\},\frac{d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{2(\kappa-\beta)}\Big) \\
&\qquad + \epsilon^{2-2\kappa}\frac{d^{-\kappa}}{2^{-\kappa}}\|U(\cdot,x)\|_{B(x,\frac{d}{2}(1+\epsilon^2))} + (\epsilon^{\kappa}+\epsilon^{2+\kappa})[U]_{\kappa,B(x,\frac{\epsilon d}{2}),\frac{\epsilon^2 d}{2}} \\
&\qquad + \epsilon^{2-\kappa} \sup_{L\leqslant \frac{\epsilon d}{2}} L^{-\kappa} \inf_l \|U(\cdot,x)-l\|_{B(x,L)}.
\end{aligned}
$$

The last term on the right-hand side can now be absorbed into the left-hand side for ϵ sufficiently small, giving the bound (3.5.14)

STEP 4. We prove that

$$\sup_{d\leqslant d_0} d^\kappa [U]_{\kappa,D_d} \lesssim \sum_{\beta\in A}\left(M^{(1)}\epsilon^{-4+2\beta-\kappa}+M^{(2)}\epsilon^{\kappa-\beta}\right)+(\epsilon^{-\kappa}+\epsilon^{2-2\kappa})\sup_{d\leqslant d_0}\|U\|_{D_d,d}. \quad (3.5.15)$$

We first argue that we can change the order of the supremum and the infimum in $\sup_{R\leqslant\frac{\epsilon d}{2}} R^{-\kappa}\inf_l \|U(\cdot,x)-l\|_{B(x,R)}$. Since $U(x,x)=0$ it is clear that one can restrict to $l(x)=0$ hence $l(y)=C(x,R).X(y-x)$. We argue that C may be chosen independently of R. Let C_R be the (near) optimal constant for the radius R. Then

$$R^{-(\kappa-1)}|C_{\frac{R}{2}}-C_R| \lesssim \sup_{R\leqslant\frac{\epsilon d}{2}} R^{-\kappa}\inf_l \|U(\cdot,x)-l\|_{B(x,R)}.$$

Since $\kappa>1$, this can be extended by summation to all $R\leqslant\frac{\epsilon d}{2}$, thus there exists a near optimal constant C independent of ρ. We then have

$$\begin{aligned}\inf_{\nu(x)}\sup_{x\neq y\in B(x,\frac{\epsilon d}{2})} d(x,y)^{-\kappa}|U(y,x)-\nu(x).X(y-x)| &\leqslant \inf_l \sup_{R\leqslant\frac{\epsilon d}{2}} R^{-\kappa}\|U(\cdot,x)-l\|_{B(x,R)}\\ &\lesssim \sup_{R\leqslant\frac{\epsilon d}{2}} R^{-\kappa}\inf_l\|U(\cdot,x)-l\|_{B(x,R)}.\end{aligned}$$

Therefore, if we take the supremum over $x\in D_d$ in Equation (3.5.14) then multiply it by d^κ and take the supremum over d, we get

$$\begin{aligned}&\sup_{d\leqslant d_0} d^\kappa \sup_{x\in D_d}\inf_{\nu(x)}\sup_{x\neq y\in B(x,\frac{\epsilon d}{2})} d(x,y)^{-\kappa}|U(y,x)-\nu(x).X(y-x)|\\ &\lesssim \sup_{d\leqslant d_0} d^\kappa\sum_{\beta\in A}\left(M^{(1)}_{D_d,\frac{d}{2}}\epsilon^{-4+2\beta-\kappa}+M^{(2)}_{D_d,\frac{d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{\kappa-\beta}\right)+\epsilon^{2-2\kappa}\sup_{d\leqslant d_0}\|U\|_{D_d,d}\\ &+\epsilon^\kappa\sup_{d\leqslant d_0} d^\kappa\sup_{x\in D_d}\sup_{y\in B(x,\frac{\epsilon d}{2})}\inf_{\nu(y)}\sup_{\substack{z\in B(y,\frac{\epsilon^2 d}{2})\\ y\neq z}} d(y,z)^{-\kappa}|U(z,y)-\nu(y).X(z-y)|.\end{aligned}$$

The last term can be absorbed into the left-hand side for ϵ small enough since for $y\in B(x,\frac{\epsilon d}{2})$ we have $d(y,\delta D)\geqslant d(1-\frac{\epsilon}{2})$ and consequently for $z\in B(y,\frac{\epsilon^2 d}{2})$, we

have $d(y,z) \leqslant \frac{\epsilon^2 d(y,\delta D)}{2(1-\frac{\epsilon}{2})} \leqslant \frac{\epsilon d(y,\delta D)}{2}$, which gives

$$\begin{aligned}
&\sup_{d\leqslant d_0} d^\kappa \sup_{x\in D_d} \inf_{\nu(x)} \sup_{x\neq y\in B(x,\frac{\epsilon d}{2})} d(x,y)^{-\kappa}|U(y,x)-\nu(x).X(y-x)| \\
&\quad\lesssim \sup_{d\leqslant d_0} d^\kappa \sum_{\beta\in A}\Big(M^{(1)}_{D_d,\frac{d}{2}}\epsilon^{-4+2\beta-\kappa} + M^{(2)}_{D_d,\frac{\epsilon d}{2},\frac{\epsilon^2 d}{2}}\epsilon^{\kappa-\beta}\Big) + \epsilon^{2-2\kappa}\sup_{d\leqslant d_0}\|U\|_{D_d,d}.
\end{aligned}$$

We concludes the proof of (3.5.15) by extending to all $y \in D_d$ with the following argument

$$\begin{aligned}
&\sup_{x\in D_d}\inf_{\nu(x)}\sup_{x\neq y\in D_d} d(x,y)^{-\kappa}|U(y,x)-\nu(x).X(y-x)| \\
&\quad\leqslant \sup_{x\in D_d}\inf_{\nu(x)}\sup_{x\neq y\in B(x,\frac{\epsilon d}{2})} d(x,y)^{-\kappa}|U(y,x)-\nu(x).X(y-x)| \\
&\qquad+\sup_{x\in D_d}\inf_{\nu(x)}\sup_{y\in D_d\setminus B(x,\frac{\epsilon d}{2})} d(x,y)^{-\kappa}|U(y,x)-\nu(x).X(y-x)| \\
&\quad\leqslant \sup_{x\in D_d}\inf_{\nu(x)}\sup_{\substack{y\in B(x,\frac{\epsilon d}{2})\\ x\neq y}} \frac{|U(y,x)-\nu(x).X(y-x)|}{d(x,y)^\kappa} + \Big(\frac{\epsilon d}{2}\Big)^{-\kappa}\|U\|_{D_d,d}.
\end{aligned}$$

Proof of Lemma 3.2.5

From the definition of $[U]_{\kappa,D}$ in (3.2.4) used with variables $x, y \in D_d$ and with triangle inequalities, we get

$$|\nu(x).X(y-x)| \leqslant [U]_{\kappa,D_d} d(x,y)^\kappa + \|U\|_{D_d,d(x,y)}.$$

Applying the interior cone condition for $r \in [0, r_d]$ gives the existence of some y with $d(x,y) = r$ such that

$$\lambda|\nu(x)|d(x,y) \leqslant [U]_{\kappa,D_d} d(x,y)^\kappa + \|U\|_{D_d,r},$$

which proves (3.2.43).

Using again the definition of $[U]_{\kappa,D}$ with variables x, y and $y, z \in D_d$, and with triangle inequalities, we get

$$\begin{aligned}
&|U(y,x)-U(z,x)-U(y,z)-(\nu(x)-\nu(z)).X(y-z)| \\
&\qquad\qquad\leqslant [U]_{\kappa,D_d}(d(x,y)^\kappa + d(y,z)^\kappa + d(x,z)^\kappa).
\end{aligned}$$

We combine this with the three-point continuity condition (3.2.38), and we assume that

$r > d(x,z) = d(y,z) \geqslant \frac{d(x,y)}{2}$ to get

$$|(\nu(x) - \nu(z) - \lambda(x,z)).X(y-z)| \lesssim d(x,z)^{\kappa}([U]_{\kappa,D} + M^{(2)}_{D_d,\frac{d}{4},\frac{d}{4}}).$$

Choosing finally y such that $|(\nu(x)-\nu(z)-\lambda(x,z)).X(y-z)| \geqslant \lambda|\nu(x)-\nu(z)||d(y,z)|$ gives (3.2.45) for $d(x,y) \leqslant r$. For $d(x,y) \geqslant r$, we have

$$d(x,y)^{-\kappa+1}|\nu(x) - \nu(y) - \lambda(x,y)| \leqslant 2r^{-\kappa+1}\|\nu\|_{D_d} + r^{-\kappa+1}|\lambda(x,y)|.$$

Applying (3.2.43) for v and (3.2.44) for λ gives for $d(x,y) \geqslant r$,

$$d(x,y)^{-\kappa+1}|\nu(x) - \nu(y) - \lambda(x,y)| \lesssim [U]_{\kappa,D_d} + r^{-\kappa}\|U\|_{D_d,r}.$$

Chapter 4

A priori bounds for the Φ^4 equation in the full sub-critical regime

4.1 Introduction

The theory of regularity structures was introduced in Hairer's groundbreaking work [41] and has since been developed into an impressive machinery [10, 11, 18] that systematically yields existence and uniqueness results for a whole range of singular stochastic partial differential equations from mathematical physics. Examples include the KPZ equation [28, 40], the multiplicative stochastic heat equation [45], as well as reversible Markovian dynamics for the Euclidean Φ^4 theory in three dimensions [41], in "fractional dimension $d < 4$" [11], for the Sine-Gordon model [20, 46], for the Brownian loop measure measure on a manifold [12] and for the $d = 3$ Yang-Mills theory [19].

A serious limitation of this theory so far is that these existence and uniqueness results only hold for a short time, and this existence time typically depends on the specific realisation of the random noise term in the equation. Most applications are furthermore limited to a compact spatial domain such as a torus. The reason for this limitation is that the whole machinery is set up as the solution theory for a mild formulation in terms of a fixed-point problem, and that specific features of the nonlinearity, such as damping effects or conserved quantities, are not taken into account. With this method, global-in-time solutions can only be obtained in special situations, e.g. if all nonlinear terms are globally Lipschitz [43] or if extra information on an invariant measure is available [22, 44].

This thesis is the beginning of a programme to derive a priori bounds within the regularity structures framework in order to go beyond short time existence and compact spatial domains. We focus on the Φ^4 dynamics which are formally given by the stochastic

reaction diffusion equation

$$(\partial_t - \Delta)\phi = -\phi^3 + \xi, \tag{4.1.1}$$

where ξ is a Gaussian space-time white noise over $\mathbb{R} \times \mathbb{R}^d$. A priori bounds for this equation have recently been derived by several groups for the two dimensional case $d = 2$ [55, 68] and the more difficult case $d = 3$ [1, 35, 36, 54]. In this last chapter we obtain bounds throughout the entire sub-critical regime, formally dealing with all "fractional dimensions" up to (but excluding) the critical dimension $d = 4$. Here we follow the convention of [11] to emulate fractional dimensions $d < 4$ by adjusting the regularity assumption on ξ, and assuming that it can only be controlled in a distributional parabolic Besov-Hölder space of regularity $-3 + \delta$ for an arbitrarily small $\delta > 0$. Connecting back to the Φ^4 dynamics driven by space-time white noise, $\delta = 0-$ mimics the scaling of the equation with $d = 4$ and $\delta = 1/2-$ gives us back equation with $d = 3$ and the result there is indeed identical to the previous chapter. This chapter shows how to deal in most generality with an unbounded (but finite) number of renormalisation terms.

Our analysis uses the method developed in the $d = 3$ context in previous chapters where it was shown that if ϕ solves (4.1.1), on a parabolic cylinder, say on

$$D = (0, 1) \times \{|x| < 1\}, \tag{4.1.2}$$

where $|x| = \max\{|x_1|, \dots, |x_d|\}$ denotes the supremum norm on $\mathbb{R}^d$, then it can be bounded on any smaller cylinder $D_R = (R^2, 1) \times \{|x| < 1 - R\}$ only in terms of the distance R and the realisation of ξ when restricted to a small neighbourhood of D. This bound holds uniformly over all possible choices for ϕ on the parabolic boundary of D, thus leveraging on the full strength of the nonlinear damping term $-\phi^3$. This makes the estimate extremely useful when studying the large scale behaviour of solutions, because given a realisation of the noise, any local function of the solution (e.g. a localised norm or testing against a compactly supported test-function) can be controlled in a *completely deterministic way* by objects that depend on the noise realisation on a compact set, without taking the behaviour of solution elsewhere into account.

Our main result is the exact analogue valid throughout the entire sub-critical regime.

Theorem 4.1.1 (Theorem 4.9.1 below)**.** *Let $\delta > 0$ and let ξ be of regularity $-3 + \delta$. Let $\{\mathbb{X}_\bullet\tau : \tau \in \mathcal{W}, \mathcal{N}\}$ be a local product lift of ξ. Let ϕ solve*

$$(\partial_t - \Delta)\phi = -\phi^{\diamond_{\mathbb{X}}3} + \xi, \qquad \text{on } D \tag{4.1.3}$$

where $\phi^{\diamond_{\mathbb{X}}3}$ refers to the renormalised cube sub-ordinate to $\mathbb{X}$.

Then $v := \phi - \sum_{\tau \in \mathcal{W}} \mathbb{X}_\bullet \mathcal{I}(\tau)$ *satisfies*

$$\|v\|_{D_R} \leqslant C \max\Big\{\frac{1}{R}, [\mathbb{X}; \tau]^{\frac{1}{\delta m_\Xi(\tau)}}, \tau \in \mathcal{N} \cup \mathcal{W}\Big\},$$

uniform in the choice of the local product, where $\| \bullet \|_{D_R}$ *denotes the supremum norm on* D_R

Here the "local product" denotes a finite number of functions/distributions $\mathbb{X}_\bullet \tau$, each of which is constructed as a polynomial of degree $m_\Xi(\tau)$, see Section 4.3. Local products correspond to *models* [41, Definition 2.17] in the theory of regularity structures, but we use them slightly differently and hence prefer a different name and notation. The functions / distributions $\mathbb{X}_\bullet \tau$ are indexed by *two* sets $\mathcal{W}$ and $\mathcal{N}$. Here $\mathcal{W}$ contains the most irregular terms so that after their subtraction the remainder v can be bounded in a positive regularity norm. The semi-norms $[\mathbb{X}; \tau]$ are defined in (4.5.11) and they correspond to the order bounds on models [41, Equation (2.15)]. The renormalised cube sub-ordinate to a local product is defined in Definition 4.7.1. This notion corresponds exactly to the reconstruction with respect to a model / local product $\mathbb{X}_\bullet$ of the abstract cube in [41].

When analysing an equation within the theory of regularity structures, one proceeds in two steps: in a *probabilistic step* a finite number of terms in a perturbative approximation of the solution are constructed - these terms are referred to as the *model* already mentioned above. The terms in this expansion are just as irregular as ϕ itself, and their construction a priori poses the same problem to define nonlinear operations. However, they are given by an explicit polynomial expression of the Gaussian noise ξ and they can thus be analysed using stochastic moment calculations. It turns out that in many situations the necessary nonlinear operations on the model can be defined despite the low regularity due to stochastic cancellations. However, this construction does require renormalisation with infinite counterterms.

In the second *analytic step* the remainder of the perturbative expansion is bounded. The key criterion for this procedure to work is a scaling condition, which is called *subcriticality* in [41], and which corresponds to *super-renormalisability* in Quantum Field Theory. This condition states, roughly speaking, that on small scales the nonlinearity is dominated by the interplay of noise and linear operator. As mentioned above, in the context of (4.1.1) this condition is satisfied precisely for $\xi \in C^{-3+\delta}$ if $\delta > 0$. Subcriticality ensures that only finitely many terms in the expansion are needed to yield a remainder that is small enough to close the argument.

It is important to note that while subcriticality ensures that the number of terms needed in the model is finite, this number can still be extremely large and typically diverges as one

approaches the threshold of criticality. A substantial part of [10, 11, 18] is thus dedicated to a systematic treatment of the algebraic relations between all of these terms and their interaction, as well as the effect of renormalising the model on the original equation. The local-in-time well posedness theory for (4.1.1) for all sub-critical $\xi \in C^{-3+\delta}$, which was developed in [11], was one of the first applications of the complete algebraic machinery.

The three dimensional analysis in the previous chapter was the first work that used regularity structures to derive a priori bounds. All of the previous works mentioned above [1, 35, 36, 54] were set in an alternative technical framework, the theory of paracontrolled distributions developed in [38]. These two theories are closely related: both theories were developed to understand the small scale behaviour of solutions to singular SPDEs, and both separate the probabilistic construction of finitely many terms in a perturbative expansion from the deterministic analysis of a remainder. Furthermore, many technical arguments in the theory of regularity structures have a close correspondent in the paracontrolled distribution framework. However, up to now paracontrolled distributions have only been used to deal with equations with a moderate number of terms in the expansion (e.g. (4.1.1) for $d \leqslant 3$ [14] or the KPZ equation [37]). Despite efforts by several groups (see e.g. [6, 7]) this method has not yet been extended to allow for expansions of arbitrary order. Thus for some of the most interesting models mentioned above, e.g. the Sine-Gordon model for β^2 just below 8π, the reversible dynamics for the Brownian loop measure on a manifold, the three-dimensional Yang-Mills theory, or the Φ^4 model close to critical dimension considered here, even a short time existence and uniqueness theory is currently out of reach of the theory of paracontrolled distributions.

The analysis developed previously is based on the idea that the large and small scale behaviour of solutions to singular SPDEs should be controlled by completely different arguments: for large scales the irregularity of ξ is essentially irrelevant and bounds follow from the strong damping effect of the nonlinearity $-\phi^3$. The small scale behaviour is controlled using the smoothing properties of the heat operator. This philosophy was implemented by working with a suitably regularised equation which could be treated with a maximum principle and by bounding the error due to the regularisation using regularity structures.

However, this analysis did not make use of the full strength of the regularity structure machinery. In fact, the three-dimensional Φ^4 equation is by now considered as one of the easiest examples of a singular SPDE, because the model only contains a moderate number of terms, only five different non-trivial products need to be defined using stochastic arguments and only two different divergencies must be renormalised. The interplay of these procedures is not too complex and no advanced algebraic machinery is needed to deal with it. Instead, we simply treated the few algebraic relations explicitly "by hand".

The main contribution of the present chapter is thus to implement a similar argument when the number of terms in the model is unbounded, thus combining the analytic ideas from [52] with the algebraic techniques of [10, 11]. For this it turns out to be most convenient to re-develop the necessary elements of the theory of regularity structures in the specific context of (4.1.1), leading to bounds that are tailor-made as input for the large-scale analysis.

Along the way, we encounter various serious simplifications and new observations which are interesting in their own right:

- As already hinted at in Theorem 4.1.1 we make systematic use of the "generalised Da Prato-Debussche trick" [11, 22]. This means that instead of working with ϕ directly we remove the most irregular terms of the expansion leading to a function valued remainder. This was already done in [11] but only in order to avoid a technical problem concerning the initial conditions. For us the remainder v is the more natural object, observing that for all values of $\delta > 0$ it solves an equation of the form

$$(\partial_t - \Delta)v = -v^3 + \dots \tag{4.1.4}$$

  where ... represents a large number of terms (the number diverges as $\delta \downarrow 0$) which involve renormalised products of either 1, v or v^2 with various irregular "stochastic terms". For each $\delta > 0$, v takes values in a positive regularity Hölder norm (i.e. it is a function) and so an un-renormalised damping term $-v^3$ appears on the right-hand side. Of course, the Hölder regularity of v is not enough to control many of the products appearing in ..., and a local expansion of v is required to control these terms. However, we are able to show that for each fixed value of δ all of these terms are ultimately of lower order relative to $(\partial_t - \Delta)v$ and v^3.

- One of the key ideas in the theory of regularity structures is *positive renormalisation* and the notion of *order*. Most of the analysis works with a re-centred version of the functions / distributions from the model, which depends on a *base-point* as well as the running argument - these objects are denoted by the Π_x. A good description of their behaviour under a change of base-point is key to the analysis, and in Hairer's framework this is accomplished by working with a family of translation operators $\Gamma_{x,y}$.

  There is a close relationship between these Π_x and $\Gamma_{x,y}$ maps and some generic identities relating them were found in [9]. Our observation is that - at least in the context of Equation (4.1.4) - most of the matrix entries for $\Gamma_{x,y}$ coincide with entries for Π_x evaluated at y. Therefore we can work with just a single object $\mathbb{X}_\bullet$

(corresponding to $\mathbf{\Pi}$ in [41]) and its re-centred version $\mathbb{X}_{\bullet,\bullet}$ that acts on itself for translation.

With this choice our framework is highly reminiscent of Gubinelli's work on branched rough paths [34], the only real difference being the introduction of some (linear) polynomials, first order derivatives, and the flexibility to allow for non-canonical products.

- Inspired by [62, 63] we work with the Schauder estimates introduced in the previous chapter using Safonov's kernel-free method (popularised in [49]), thus working directly with the PDE rather than transforming into an integral equation. This is more convenient for our analysis, because these bounds give more flexibility e.g. when localising functions by restricting them to certain sets.
- As in [41] we use the model / local product to build a local approximation of v around any base-point x. This takes the form

$$v(y) \approx \sum_{\tau \in \mathcal{N}} \Upsilon_x(\tau) \mathbb{X}_{y,x} \mathcal{I}(\tau),$$

with a well-controlled error as y approaches x. In order to use this local expansion to control nonlinearities two key analytic ingredients are needed: the first is the order bound discussed above, and the second is a suitable continuity condition on the coefficients $\Upsilon_x(\tau)$. In [41] these conditions are encoded in a family of model-dependent semi-norms, which make up the core of the definition of a *modelled distribution* [41, Definition 3.1]. It turns out however, that the coefficients $\Upsilon_x(\tau)$ that appear in the expansion of the solution v are far from generic: up to signs and combinatoric factors they can only be either 1, $v(x)$, $v(x)^2$, or $v_{\mathbf{X}}(x)$ (a generalised derivative of v). Furthermore, there is a simple criterion (Lemma 4.6.7) to see which of these is associated to a given tree τ. This fact was already observed in [11] and was called *coherence* there. Here we observe that the various semi-norms in the definition of a modelled distribution are in fact all truncations of the single continuity condition on the first coefficient $\Upsilon(\mathbf{1}) = v$. This observation is key for our analysis, as this particular semi-norm is precisely the output of our Schauder lemma.
- Our deterministic theory more cleanly separates the issues of positive and negative renormalisation in the context of (4.1.1). Indeed, we can derive a priori bounds under extremely general assumptions on the specific choice of the local product $\mathbb{X}$ which seems quite a bit larger and simpler than the space of models given in [10]. The key information contained in $\mathbb{X}$ is how certain a priori unbounded products should be interpreted. Our definition of a local product allows for these

interpretations to be completely arbitrary! We can then always define the re-centred version of $\mathbb{X}$ (or path) and the only assumption where the various functions interact is in the assumption that these re-centred products satisfy the correct order bound.

We do however include a Section 4.8 in which we introduce a specific class of local products for which the renormalised product $\phi^{\diamond_{\mathbb{X}}3}$ appearing in (4.1.3) is still a local polynomial in ϕ and its spatial derivatives. Our approach in this section is to apply a recursive negative renormalisation that commutes with positive renormalisation, similar to [9]. Finally, the class of local products described in Section 4.8 also contains local products that correspond to the BPHZ renormalised model [10, 18].

4.1.1 Conventions

Throughout we will work with functions / distributions defined on (subsets of) $\mathbb{R} \times \mathbb{R}^d$ for an arbitrary $d \geqslant 1$. As in previous chapters, we measure regularity in Hölder-type norms that reflect the parabolic scaling of the heat operator. We recall the definition of α-Hölder norm for $\alpha < 0$ we set

$$[\xi]_\alpha = \sup_{L \leqslant 1} \Big\| (\xi)_L \Big\| L^{-\alpha}. \tag{4.1.5}$$

The operator $(\bullet)_L$ denotes convolution with a compactly supported smooth kernel $\Psi_L(x) = L^{-d-2}\Psi\Big(\frac{x_0}{L^2}, \frac{\overline{x}}{L}\Big)$, where $x = (x_0, \overline{x})$. As before, we work with a specific choice of Ψ, but this was only relevant in the proof of the Reconstruction Theorem, Lemma 3.2.8.

In the case of space-time white noise, the quantity in (4.1.5) is almost surely not finite, but our analysis only depends on the noise locally: a space-time cutoff can be introduced. Throughout the paper we also make the *qualitative* assumption that ξ and all other functions are smooth. This corresponds to introducing a regularisation of the noise term ξ (e.g. by convolution with a regularising kernel at some small scale - in field theory this is called an ultra-violet cutoff). This is very convenient, because it allows to avoid unnecessary discussions about how certain objects have to be interpreted and in which sense partial differential equations hold. We stress however that our main result, Theorem 4.9.1, is a bound only in terms of those low-regularity norms (Definition 4.5.7) which can be controlled when the regularisation is removed in the renormalization procedure. Even though all functions involved are smooth, we will freely use the term "distribution" to refer to a smooth function that can only be bounded in a negative regularity norm.

4.2 Overview

As stated in the introduction a large part of our analysis consists of a suitable reformulation of elements of the theory of regularity structures. The key notions we require are *local products*, the *renormalized product* sub-ordinate to a local product, as well as the relevant norms that permit us to bound these renormalized products. We start our exposition with an overview over these notions and how they are interconnected. The exposition in this section is meant to be intuitive and rather "bottom up". The actual analysis begins in the subsequent Section 4.3.

4.2.1 Subcriticality:

The starting point of our analysis is a simple scaling consideration: assume ϕ solves

$$(\partial_t - \Delta)\phi = -\phi^3 + \xi, \tag{4.2.1}$$

for $\xi \in C^{-3+\delta}$. Schauder theory suggests that the solution ϕ is not better than $C^{-1+\delta}$. In this low regularity class no bounds on ϕ^3 are available, but as we will see below the notion of product we will work with has the property that negative regularities add under multiplication. Therefore we will obtain a control on (a renormalised version of) ϕ^3 as a distribution in $C^{-3+3\delta}$. Despite this very low regularity, for $\delta > 0$, the term ϕ^3 is still more regular than the noise ξ. This observation is the core of Hairer's notion of sub-criticality (see [41, Assumption 8.3]) and suggests that the small-scale behaviour of ϕ and ϕ^3 can ultimately be well understood by building a perturbative expansion based on the linearised equation.

4.2.2 Trees:

We follow Hairer's convention to index the terms in this expansion by a set of trees. This is not only a convenient notation that organises which term corresponds to which operation, but also allows for an efficient organisation of the relations between these terms. We furthermore follow the convention to view trees as abstract symbols which form the basis of a finite-dimensional vector space. The trees are built from a generator symbol Ξ (which represents the noise ξ and graphically are the leaves of the tree) followed by applying the operator $\mathcal{I}(\cdot)$ (which represents to solving the heat equation and graphically corresponds to the edges of the tree) and taking products of trees (which represents to some choice of point-wise product and graphically corresponds to joining two trees at their root). To carry out the localisation procedure, discussed in Section 4.2.4 below, along with Ξ, additional generators $\{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$ are used in our construction of trees.

We associate concrete meaning to trees via an operator $\mathbb{X}_\bullet$ of one variable which we call a "local product", see Definition 4.3.7. Even though this may seem somewhat bulky initially, it turns out to be extremely convenient as the concrete definition of $\mathbb{X}_\bullet$ on the same tree may change during the renormalisation procedure and because, the local product also appears in a *centred form* with two variables, denoted by $\mathbb{X}_{\bullet,\bullet}$, see Section 4.2.6 below.

4.2.3 Subtracting the most irregular terms:

The first step of our analysis consists of subtracting a finite number of terms from ϕ to obtain a remainder v which is regular enough to be bounded in a positive Hölder norm. The regularity analysis in Section 4.2.1 suggests that the regularity of ϕ can be improved by removing ξ from the right-hand side of (4.2.1). We introduce the first graph, $\mathcal{I}(\Xi)$ or graphically $\treeI$, and impose that $\mathbb{X}_\bullet$ acts on this symbol yielding a function that satisfies

$$(\partial_t - \Delta)\mathbb{X}_\bullet\treeI = \xi. \tag{4.2.2}$$

We set $\widetilde{v} := \phi - \mathbb{X}_\bullet\treeI$ so that $\widetilde{v}$ solves

$$(\partial_t - \Delta)\widetilde{v} = -\phi^3 = -\big(\widetilde{v}^3 + 3\widetilde{v}^2\mathbb{X}_\bullet\treeI + 3\widetilde{v}\big(\mathbb{X}_\bullet\treeI\big)^2 + \big(\mathbb{X}_\bullet\treeI\big)^3\big). \tag{4.2.3}$$

Of course the problem of controlling the cube of a distribution of regularity $-1+\delta$ has not disappeared, but instead of ϕ^3 one now has to control $(\mathbb{X}_\bullet\treeI)^3$ and $(\mathbb{X}_\bullet\treeI)^2$. At this point one has to make use of the fact that $\mathbb{X}_\bullet\treeI$ is known much more explicitly than the solution ϕ, and can thus be analysed using explicit covariance calculations. We do not discuss these calculations here, but rather view these products as part of the given data: we introduce two additional symbols $\mathcal{I}(\Xi)\mathcal{I}(\Xi)\mathcal{I}(\Xi)$ or graphically $\treeIIIa$, and similarly $\mathcal{I}(\Xi)\mathcal{I}(\Xi)$ or $\treeIIa$ and assume that $\mathbb{X}$ acts on these additional symbols yielding distributions which are controlled in $C^{-3+3\delta}$ and $C^{-2+2\delta}$. We stress that only the control on these norms enters the proof of our a priori bound, and no relation to $\mathbb{X}_\bullet\treeI$ needs to be imposed (see however Section 4.8 below). Instead of (4.2.3) we thus consider

$$(\partial_t - \Delta)\widetilde{v} = -\big(\widetilde{v}^3 + 3\widetilde{v}^2\mathbb{X}_\bullet\treeI + 3\widetilde{v}\mathbb{X}_\bullet\treeIIa + \mathbb{X}_\bullet\treeIIIa\big). \tag{4.2.4}$$

Note that the most irregular term on the right-hand side is $\mathbb{X}_\bullet\treeIIIa \in C^{-3+3\delta}$ so that we can expect $\widetilde{v} \in C^{-1+3\delta}$ i.e. we have gained 2δ differentiability with respect to ϕ. For $\delta > \frac{1}{3}$ (which corresponds to dimensions "$d < 3\frac{1}{3}$") $\widetilde{v}$ is thus controlled in a positive order Hölder norm. For smaller δ we proceed to subtract an additional term to again remove the most irregular term from the right-hand side as above. We define a new symbol $\mathcal{I}(\mathcal{I}(\Xi)\mathcal{I}(\Xi)\mathcal{I}(\Xi))$ or graphically $\treeIVa$, postulate that $\mathbb{X}_\bullet$ acts on this symbol yielding a

distribution which solves

$$(\partial_t - \Delta)\mathbb{X}_\bullet \Upsilon = \mathbb{X}_\bullet \Psi, \tag{4.2.5}$$

and define a new remainder $\widetilde{\widetilde{v}} := \widetilde{v} + \mathbb{X}_\bullet \Upsilon = \phi - \mathbb{X}_\bullet \uparrow + \mathbb{X}_\bullet \Upsilon$ which takes values in $C^{-1+5\delta}$. In general, for any $\delta > 0$ we denote by $\mathcal{W}$ the set of trees of *order* < -2 (for these trees, order is the same as the regularity of the local product on this tree; below, in Section 4.2.6 we will encounter additional trees for which these notions differ) and define

$$v := \phi - \sum_{w \in \mathcal{W}} (-1)^{\frac{m(w)-1}{2}} \mathbb{X}_\bullet \mathcal{I}(\tau),$$

where $m(w)$ denotes the number of "leaves" of the tree w. Then v takes values in a Hölder space of positive regularity. The remainder equation then turns into

$$\begin{aligned}(\partial_t - \Delta)v = &-v^3 \\ &- 3 \sum_{w \in \mathcal{W}} (-1)^{\frac{m(w)-1}{2}} v^2 \mathbb{X}_\bullet \mathcal{I}(w) \\ &- 3 \sum_{w_1, w_2 \in \mathcal{W}} (-1)^{\frac{m(w_1)+m(w_2)-2}{2}} v \mathbb{X}_\bullet (\mathcal{I}(w_1)\mathcal{I}(w_2)) \\ &- \sum_{\substack{w_1, w_2, w_3 \in \mathcal{W} \\ \mathcal{I}(w_1)\mathcal{I}(w_2)\mathcal{I}(w_3) \notin \mathcal{W}}} (-1)^{\frac{m(w_1)+m(w_2)+m(w_3)-3}{2}} \mathbb{X}_\bullet (\mathcal{I}(w_1)\mathcal{I}(w_2)\mathcal{I}(w_3)).\end{aligned} \tag{4.2.6}$$

We stress that the structure of this equation is always the same in the sense that the nonlinear heat equation $(\partial_t - \Delta)v = -v^3$ is perturbed by a large number of irregular terms (the number actually diverges as $\delta \to 0$). Bounding these irregular terms forces us to introduce additional trees as we will see below, but ultimately we will show that all of these terms are of lower order with respect to $(\partial_t - \Delta)v = -v^3$.

4.2.4 Iterated freezing of coefficients

We now discuss the remainder equation (4.2.6) in more detail, writing it as

$$(\partial_t - \Delta)v = -v^3 - 3v^2 \mathbb{X}_\bullet \uparrow - 3v \mathbb{X}_\bullet \Psi - \Upsilon(\tau_0)\mathbb{X}_\bullet \tau_0 - \dots, \tag{4.2.7}$$

where we are isolating the most irregular terms in each of the three sums appearing on the right-hand side of (4.2.6). The most irregular term in the sum on the second line of (4.2.6) is $-3v^2\mathbb{X}_\bullet \uparrow$ and the most irregular term in the third line is $-3v\mathbb{X}_\bullet \Psi$. For the last line, the precise form of the most irregular term depends on δ and there could be multiple terms of the same low regularity. Here we just keep track of one of them, simply denote it by $\mathbb{X}_\bullet \tau_0$ and also leave the combinatorial prefactor $\Upsilon(\tau_0)$ implicit. We remark that $\mathbb{X}_\bullet \tau_0$ is always a distribution of regularity $C^{-2+\kappa}$ for some $\kappa \in (0, 2\delta)$. To simplify the

exposition we disregard all of the (many) additional terms hidden in the ellipses ... for the moment.

We recall the standard multiplicative inequality

$$\|fg\|_{C^{-\beta}} \lesssim \|f\|_{C^{\alpha}} \|g\|_{C^{-\beta}}$$

for $\alpha, \beta > 0$ which holds if and only if $\alpha - \beta > 0$. In view of the regularity $\mathbb{X}_\bullet \mathbb{V} \in C^{-2+2\delta}$ we would thus require $v \in C^\gamma$ for $\gamma > 2 - 2\delta$ in order to have a classical interpretation of the product $v\mathbb{X}_\bullet \mathbb{V}$ on the right-hand side of (4.2.7). Unfortunately, v is much more irregular: by Schauder theory we can only expect v to be of class C^κ.

The solution to overcome this difficulty presented in [41] amounts to an "iterated freezing of coefficient" procedure to obtain a good local description of v around a fixed base-point: we fix a space-time point x and rewrite the third (and most important) term on the right-hand side of (4.2.7) as

$$v\mathbb{X}_\bullet \mathbb{V} = v(x)\mathbb{X}_\bullet \mathbb{V} + (v - v(x))\mathbb{X}_\bullet \mathbb{V} \tag{4.2.8}$$

and use this to rewrite the equation (4.2.7) as

$$\begin{aligned}(\partial_t - \Delta)(v + 3v(x)\mathbb{X}_\bullet \mathbb{Y} + \Upsilon(\tau_0)\mathbb{X}_\bullet \mathcal{I}(\tau_0)) \\ = -v^3 - 3v^2\mathbb{X}_\bullet \mathbb{I} - 3(v - v(x))\mathbb{X}_\bullet \mathbb{V} - \dots\end{aligned} \tag{4.2.9}$$

where we have introduced new symbols $\mathbb{Y}$ and $\mathcal{I}(\tau_0)$ and postulated that $\mathbb{X}$ acts on these symbols to yield a solution of the inhomogeneous heat equation with right-hand sides $\mathbb{X}_\bullet \mathbb{V}$ and $\mathbb{X}_\bullet \tau_0$. The worst term on the right-hand side is now $\mathbb{X}_\bullet \mathbb{V}$ so that the left-hand side can at best be of regularity 2δ. However, near the base-point we can use the smallness of the pre-factor $|v(\bullet) - v(x)| \lesssim [v]_\kappa d(\bullet, x)^\kappa$ to get the better estimate

$$\begin{aligned}|U(y,x)| &:= \Big| v(y) - \big(v(x) - 3v(x)\mathbb{X}_{y,x}\mathbb{Y} - \Upsilon(\tau_0)\mathbb{X}_{y,x}\mathcal{I}(\tau_0)\big)\Big| \\ &\lesssim d(y,x)^{2\delta+\kappa},\end{aligned} \tag{4.2.10}$$

where have used the short-hand notation

$$\begin{aligned}\mathbb{X}_{y,x}\mathbb{Y} &:= \mathbb{X}_y \mathbb{Y} - \mathbb{X}_x \mathbb{Y} \\ \mathbb{X}_{y,x}\mathcal{I}(\tau_0) &:= \mathbb{X}_y \mathcal{I}(\tau_0) - \mathbb{X}_x \mathcal{I}(\tau_0).\end{aligned} \tag{4.2.11}$$

This bound in turn can now be used to get yet a better approximation in (4.2.8): we write

$$(v(y) - v(x))\mathbb{X}_y \mathbb{V}$$

$$= \big(U(y,x) - 3v(x)\mathbb{X}_{y,x}\Upsilon - \Upsilon(\tau_0)\mathbb{X}_{y,x}\mathcal{I}(\tau_0)\big)\mathbb{X}_y\Upsilon. \tag{4.2.12}$$

At this point two additional non-classical products appear in the second and third term on the right-hand side, and as before they are treated as part of the assumed data: we introduce two additional symbols Υ and $\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi)$ and assume that $\mathbb{X}$ acts on these symbols yielding distributions which we interpret as playing the roles of the products $\mathbb{X}_y\Upsilon\mathbb{X}_y\Upsilon$ and $\mathbb{X}_y\mathcal{I}(\tau_0)\mathbb{X}_y\Upsilon$. Similarly, we introduce the base-point dependent versions as

$$\begin{aligned}\mathbb{X}_{y,x}\Upsilon &:= \mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon\mathbb{X}_y\Upsilon\\ \mathbb{X}_{y,x}\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi) &:= \mathbb{X}_y\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi) - \mathbb{X}_x\mathcal{I}(\tau_0)\mathbb{X}_y\Upsilon,\end{aligned} \tag{4.2.13}$$

so that (4.2.12) becomes re-interpreted as

$$\begin{aligned}&(v(y) - v(x))\mathbb{X}_y\Upsilon\\ &= U(y,x)\mathbb{X}_y\Upsilon - 3v(x)\mathbb{X}_{y,x}\Upsilon - \Upsilon(\tau_0)\mathbb{X}_{y,x}\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi).\end{aligned} \tag{4.2.14}$$

The last two terms on the right-hand side can now again be moved to the left-hand side of the equation suggesting that near x we can improve the approximation (4.2.9) of $v(y)$ by considering

$$\widetilde{U}(y,x) := U(y,x) + 3v(x)\mathbb{X}_{y,x}\Upsilon + \Upsilon(\tau_0)\mathbb{X}_{y,x}\mathcal{I}(\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi)) \tag{4.2.15}$$

where

$$\begin{aligned}\mathbb{X}_{y,x}\Upsilon &:= \mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon - \mathbb{X}_x\Upsilon\big(\mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon\big)\\ \mathbb{X}_{y,x}\mathbb{X}_y\mathcal{I}(\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi)) &:= \mathbb{X}_y\mathcal{I}(\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi)) - \mathbb{X}_x\mathcal{I}(\mathcal{I}(\tau_0)\mathcal{I}(\Xi)\mathcal{I}(\Xi))\\ &\qquad - \mathbb{X}_x\mathcal{I}(\tau_0)\big(\mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon\big).\end{aligned} \tag{4.2.16}$$

with the improved estimate $|\widetilde{U}(y,x)| \lesssim d(y,x)^{4\delta+\kappa}$, thus gaining another 2δ with respect to $U(y,x)$.

The whole procedure can now be iterated: in each step an improved approximation of v is plugged into the product $v\mathbb{X}_\bullet\Upsilon$ which in turn yields an even better local approximation of v near x. At some point additional terms have to be added:

- In order to get a local description of order > 1, "generalized derivatives" $v_{\mathbf{X}_i}$ of v appears, i.e. a term $\sum_{i=1}^d v_{\mathbf{X}_i}(x)(y_i - x_i)$ has to be included.
- The term $-3v^2\mathbb{X}\Upsilon$ on the right-hand side of the remainder equation (4.2.7) has regularity $-1+\delta$, so once one wishes to push the expansion of v to a level $> 1+\delta$,

one also has to "freeze the coefficient" v^2, i.e. write

$$\begin{aligned} v^2\mathbb{X}\text{🙾} &= v^2(x)\mathbb{X}\text{🙾} + (v^2 - v^2(x))\mathbb{X}\text{🙾} \\ &= v^2(x)\mathbb{X}\text{🙾} + 2v\big(- 3v(x)\mathbb{X}_{\bullet,x}\text{🙿} - \Upsilon(\tau_0)\mathbb{X}_{\bullet,x}\mathcal{I}(\tau_0)\big)\mathbb{X}\text{🙾} + \dots \end{aligned}$$

leading to additional terms on the left-hand side.

- Of course, the various terms which were hidden in ... in (4.2.7) above have to be treated in a similar way leading to (many) additional terms in the local description of v.

Ultimately, we iterate this scheme until we have a local description of order $\gamma > 2 - 2\delta$, corresponding to the regularity required classically to define $v\mathbb{X}_\bullet\text{🙿}$.

4.2.5 Renormalised products

The previous discussion thus suggests that we have a Taylor-like approximation of v near the base-point x

$$v(y) \approx v(x) + \sum_{i=1}^{d} v_{\mathbf{X}_i}(x) \cdot (y_i - x_i) + \sum_{\tau \in \mathring{\mathcal{N}}} \Upsilon_x(\tau)\mathbb{X}_{y,x}\mathcal{I}(\tau) \tag{4.2.17}$$

for coefficients Υ_x and with an error that is controlled by $\lesssim d(x,y)^\gamma$. Here $\mathring{\mathcal{N}}$ denotes the set of trees appearing in the recursive construction described above. We unify our notation by also writing the first two terms with "trees" and set

$$\begin{aligned} \mathbb{X}_{y,x}\mathcal{I}(\mathbf{1}) &= 1 & \mathbb{X}_{y,x}\mathcal{I}(\mathbf{X}_i) &= y_i - x_i \\ \Upsilon_x(\mathbf{1}) &= v(x) & \Upsilon_x(\mathbf{X}_i) &= v_{\mathbf{X}_i}(x), \end{aligned}$$

thus permitting us to rewrite (4.2.17) as

$$v(y) \approx \sum_{\tau \in \mathcal{N}} \Upsilon_x(\tau)\mathbb{X}_{y,x}\mathcal{I}(\tau), \tag{4.2.18}$$

where $\mathcal{N} = \mathring{\mathcal{N}} \cup \{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$.

Of course, up to now our reasoning was purely formal, because it relied on all of the ad hoc products of singular distributions that were simply postulated along the way. We now turn this formal reasoning into a *definition* of the products subordinate to the choices in the local product $\mathbb{X}$. More precisely, we *define* renormalized products such as

$$v \circ_{\mathbb{X}} \mathbb{X}_\bullet\text{🙿}(x) := \sum_{\tau \in \mathcal{N}} \Upsilon_x(\tau)\mathbb{X}_{x,x}(\mathcal{I}(\tau)\text{🙿}),$$

$$v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \text{\Large\textvisiblespace}(x) := \sum_{\tau_1,\tau_2 \in \mathcal{N}} \Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\mathbb{X}_{x,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_1)\text{\Large\textvisiblespace}). \tag{4.2.19}$$

Our main a priori bound in Theorem 4.9.1 holds for the remainder equation interpreted in this sense, under very general assumptions on the local product $\mathbb{X}$. However, under these very general assumptions it is not clear (and in general not true) that the renormalized products are in any simple relationship to the *usual* products. In Section 4.8 we discuss a class of local products for which the renormalized products can be re-expressed as explicit local functionals of *usual* products. In particular, for those local products we always have

$$\phi^{\circ_{\mathbb{X}} 3}(y) = \phi^3(y) - a\phi^2(y) - b\phi(y) - c - \sum_{i=1}^{d} d_i \partial_i \phi(y),$$

for real parameters a, b, c, d_i. This class of local products contains the examples that can actually be treated using probabilistic arguments.

4.2.6 Positive renormalisation and order

One of the key insights of the theory of regularity structures is that the renormalized products defined above can be controlled quantitatively in a process called renormalization, and the most important ingredient for that process are the definitions of suitable notions of regularity / continuity for the local products $\mathbb{X}$ and the coefficients Υ. We start with the local products.

The base-point dependent or centred versions of the local product, $\mathbb{X}_{y,x}$ that appear naturally in the expansions above (e.g. in (4.2.11), (4.2.13), (4.2.16)) are in fact much more than a notational convenience. The key observation is that their behaviour as the running argument y approaches the base-point x is well controlled in the so-called *order bound*. For $\mathbb{X}_{y,x}\Upsilon$ defined in (4.2.11) we have

$$|\mathbb{X}_{y,x}\Upsilon| = |\mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon| \lesssim d(y,x)^{2\delta}, \tag{4.2.20}$$

which amounts to the Hölder regularity of $\mathbb{X}_\bullet\Upsilon$. The order bounds become more interesting in more complex examples: for $\mathbb{X}_{y,x}\Upsilon^{\bullet\bullet}$ defined in (4.2.16) we have

$$|\mathbb{X}_{y,x}\Upsilon^{\bullet\bullet}| := \big|\mathbb{X}_y\Upsilon^{\bullet\bullet} - \mathbb{X}_x\Upsilon^{\bullet\bullet} - \mathbb{X}_x\Upsilon\big(\mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon\big)\big| \lesssim d(y,x)^{4\delta}. \tag{4.2.21}$$

The remarkable observation here is that the function $\mathbb{X}_y\Upsilon^{\bullet\bullet}$ is itself only of regularity 2δ, so that this estimate expresses that the second term $-\mathbb{X}_x\Upsilon\big(\mathbb{X}_y\Upsilon - \mathbb{X}_x\Upsilon\big)$ exactly compensates the roughest small scale fluctuations. The exponent 4δ is defined as the *order* of the tree $\Upsilon^{\bullet\bullet}$ simply denoted by $|\Upsilon^{\bullet\bullet}|$. Analogously, for the tree $\mathbb{X}_{y,x}\Upsilon^{\bullet\bullet}$ defined in

(4.2.13) we have the order $|\Xi| = -2 + 4\delta$ exceding the *regularity* of the distribution $\mathbb{X}_y\Xi$ which is only $-2 + 2\delta$, the same as the regularity of $\mathbb{X}_y\Upsilon$. As these quantities are distributions the *order* bound now has to be interpreted by testing against the rescaled kernel Ψ_T

$$\Big| \int \Psi_T(y - x)\mathbb{X}_{y,x}\Xi \; dy \Big| \lesssim T^{-2+4\delta}. \tag{4.2.22}$$

This notion of order of trees has the crucial property that it behaves additively under multiplication - just like the regularity of distributions discussed above. This property is what guarantees that for sub-critical equations the number of trees with order below any fixed threshold is always finite.

4.2.7 Change of base-point

As sketched in the discussion above, the base-point dependent centred local products $\mathbb{X}_{\bullet,\bullet}$ are defined recursively from the un-centred ones. For what follows, a good algebraic framework to describe the centring operation and the behaviour under the change of base-point is required. It turns out that both operations can be formulated conveniently using a combinatorial operation called the *coproduct* Δ (note that this Δ has nothing to do with the Laplace operator, it will always be clear from the context which object we refer to). This coproduct associates to each tree a finite sum of couples $(\tau^{(1)}, \tau^{(2)})$ where $\tau^{(1)}$ is a tree and $\tau^{(2)}$ is a finite list of trees. Equivalently, the coproduct can be seen as a linear map

$$\Delta : \mathcal{T}_+ \to \mathrm{Vec}(\mathcal{T}_+) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}}),$$

where $\mathcal{T}_+$ and $\mathcal{T}^{\mathrm{rec}}$ are sets of trees that we will define later (see Section 4.3.1 for the former and Section 4.4.3 for the latter) and Vec and Alg denote the vector space and the free non-commutative unital algebra generated by a set, respectively. This coproduct is defined recursively reflecting exactly the recursive positive renormalization described above in Section 4.2.4. For example

$$\begin{aligned}
\Delta\Upsilon &:= \Upsilon \otimes \mathcal{I}(\mathbf{1}) + \mathcal{I}(\mathbf{1}) \otimes \Upsilon \\
\Delta\Xi &:= \Xi \otimes \mathcal{I}(\mathbf{1}) + \mathcal{I}(\mathbf{1}) \otimes \Xi + \Upsilon \otimes \Upsilon
\end{aligned}$$

so that for example the first definitions of (4.2.11) and (4.2.15) turn into

$$\begin{aligned}
\mathbb{X}_{y,x}\Upsilon &:= (\mathbb{X}_y \otimes \mathbb{X}_x^{\mathrm{rec}})\Delta\Upsilon \\
\mathbb{X}_{y,x}\Xi &:= (\mathbb{X}_y \otimes \mathbb{X}_x^{\mathrm{rec}})\Delta\Xi,
\end{aligned}$$

i.e. the different terms in the coproduct correspond to the different terms appearing in the positive renormalization, and for each pair $\tau^1 \otimes \tau^2$, the first tree τ^1 corresponds to the "running variable y" and τ^2 to the value of the base-point. The coefficients $\mathbb{X}_x^{\mathrm{rec}}$ are also defined recursively to match this definition e.g

$$\mathbb{X}_x^{\mathrm{rec}}\mathcal{I}(\mathbf{1}) = 1 \qquad \mathbb{X}_x^{\mathrm{rec}}\Upsilon = -\mathbb{X}_x\Upsilon \qquad \mathbb{X}_x^{\mathrm{rec}}\Upsilon^{\bullet\bullet} = -\mathbb{X}_x\Upsilon^{\bullet\bullet} + \mathbb{X}_x\Upsilon\mathbb{X}_x\Upsilon.$$

This way of codifying the relation between the centred and un-centred local products is useful, e.g. when analysing the effect of the renormalization procedure (Section 4.8) but even more importantly they give an efficient way to describe how $\mathbb{X}_{y,x}$ behaves under change of base-point. It turns out that we obtain the remarkable formula for all $\tau \in \mathcal{T}$

$$\mathbb{X}_{y,z}(\tau) = (\mathbb{X}_{y,\bar{z}} \otimes \mathbb{X}_{\bar{z},z})\Delta\tau,$$

i.e. the centred object $\mathbb{X}_{y,z}$ acts on itself as a translation operator!

4.2.8 Continuity of coefficients

With this algebraic formalism in hand, we are now ready to describe the correct continuity condition on the coefficients. This continuity condition is formulated in terms of the concrete realisation of the local product, in that an "adjoint" of the translation operator appears. In order to formulate it, we introduce another combinatorial notation $C_+(\overline{\tau}, \tau)$, which is defined recursively to ensure that

$$\Delta\mathcal{I}(\tau) = \sum_{\overline{\tau} \in \mathcal{N} \cup \mathcal{W}} \mathcal{I}(\overline{\tau}) \otimes C_+(\overline{\tau}, \tau).$$

We argue below that the correct family of semi-norms for the various coefficients $\Upsilon(\tau)$ is given by

$$\sup_{x,y} \frac{1}{d(x,y)^{\gamma - |\tau|}} \Big| \Upsilon_x(\tau) - \sum_{\substack{\overline{\tau} \in \mathcal{N} \\ |\overline{\tau}| < \gamma}} \Upsilon_y(\overline{\tau}) \mathbb{X}_{y,x} C_+(\tau, \overline{\tau}) \Big|. \tag{4.2.23}$$

The Reconstruction Theorem (see Lemma 3.2.8 for our formulation) implies that the renormalized products (4.2.19) can be controlled in terms of the semi-norms (4.2.23) and the order bounds (e.g. (4.2.22)). Reconstruction takes as input the whole family of semi-norms (4.2.23), but it turns out that in our case, it suffices to deal with a single semi-norm on the coefficients: the coefficients $\Upsilon_x(\tau)$ that appear in the recursive freezing of coefficients described in Section 4.2.4 are far from arbitrary. It is very easy to see that (up to combinatorial coefficients and signs) the only possible coefficients we encounter are v, v^2, $v_{\mathbf{X}}$, and 1. It then turns out that all of the semi-norms (4.2.23) are in fact truncations of the single continuity condition on the coefficient v itself. This semi-norm

can then be easily seen to be

$$\sup_{x,y} \frac{1}{d(x,y)^\gamma} \Big| v(x) - \sum_{\substack{\tau \in \mathcal{N} \\ |\tau| < \gamma}} \Upsilon_y(\tau) \mathbb{X}_{y,x} \mathcal{I}(\tau) \Big|, \tag{4.2.24}$$

which measures precisely the quality of the approximation (4.2.18) at the starting point of this discussion.

4.2.9 Outline of chapter

A large part of this chapter is concerned with providing the details of the arguments sketched above in a streamlined "top-down" way: The set of trees, their order and local products are defined in Section 4.3, while Section 4.4 provides a systematic treatment of combinatorial properties of the coproduct. Positive renormalization of local products is discussed in Section 4.5, while Section 4.6 contains the detailed discussion of the coefficients Υ sketched above in Section 4.2.5. The renormalized products in the spirit of (4.2.19) are defined in Section 4.7. As already announced above Section 4.8 contains the discussion of a special class of local products, for which the renormalized product can be expressed in a simple form. The actual large-scale analysis only starts in Section 4.9, where the main result is announced. This section also contains a detailed outline of the strategy of proof. The various technical lemmas that constitute this proof can then be found in Section 4.10.

4.3 Tree expansion and local products

The objects we refer to as trees will be built from

- a set of generators $\{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d, \Xi\}$, which can be thought of as the set of possible types of leaf nodes of the tree
- applications of an operator $\mathcal{I}$, which can be thought of as edges
- A tree product, which joins edges $\mathcal{I}$ at a common new node.

As an example, we have

$$\Xi = \bullet,\ \mathcal{I}[\Xi]^2 = \vee,\ \mathcal{I}(\Xi)\mathcal{I}(\mathcal{I}(\Xi))\mathcal{I}(\Xi) =$$

,

$$\mathcal{I}(\Xi)\mathcal{I}(\mathcal{I}(\Xi)^2\mathcal{I}(\mathbf{X}_i))\mathcal{I}(\Xi) =$$

, and $\mathcal{I}(\mathbf{1})\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\mathbf{1}) =$

.

In particular, when drawing our trees pictorially we decorate the leaf nodes with a $\bullet$ for an instance of Ξ, 0 for an instance of $\mathbf{1}$, and $j \in \{1, \dots, d\}$ for an instance of $\mathbf{X}_j$. Notice

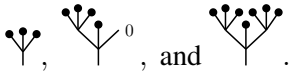
that we do not decorate internal (nonleaf) nodes and have the root node at the bottom.

Our tree product is non-commutative which in terms of our pictures means that we distinguish between the ways a tree can be embedded in the plane. For example, the following trees are treated as distinct from the trees above:

$$\mathcal{I}(\Xi)\mathcal{I}(\mathcal{I}(\Xi)\mathcal{I}(\mathbf{X}_i)\mathcal{I}(\Xi))\mathcal{I}(\Xi) = \text{  } \text{ and } \mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\mathbf{1})\mathcal{I}(\mathbf{1}) = \text{  } .$$

Remark 4.3.1. We work with a non-commutative tree product only to simplify combinatorial arguments. Whenever we map trees over to concrete functions and or distributions this mapping will treat identically any two trees that coincide when one imposes commutativity of the tree product.

We say a tree τ is planted if it is of the form $\tau = \mathcal{I}(\widetilde{\tau})$ for some other tree $\widetilde{\tau}$, some examples would be:



We take a moment to describe the intuition behind these trees. The symbol Ξ will represent the driving noise, we will often call nodes of type Ξ noise leaves/nodes. Regarding the operator $\mathcal{I}$, when applied to trees different from $\{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$, $\mathcal{I}$ will represent solving the heat equation, that is

$$\text{“}(\partial_t - \Delta)\mathcal{I}(\tau) = \tau\text{”} .$$

However, we think of the trees as algebraic objects so such an equation is only given here as a mnemonic and will be made concrete when we associate functions to trees in Section 4.3.2.

The symbols $\{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$ themselves will not correspond to any analytic object, but the trees $\{\mathcal{I}(\mathbf{1}), \mathcal{I}(\mathbf{X}_1), \dots, \mathcal{I}(\mathbf{X}_d)\}$ will play the role of the classical monomials, that is $\mathcal{I}(\mathbf{1})$ corresponds to 1 and $\mathcal{I}(\mathbf{X}_j)$ corresponds to the monomial z_j.

We define $\widehat{\mathcal{T}}_r$ to be the smallest set of trees containing $\{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d, \Xi\} \subset \widehat{\mathcal{T}}_r$ and such that for every $\tau_1, \tau_2, \tau_3 \in \widehat{\mathcal{T}}_r$ one also has $\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \widehat{\mathcal{T}}_r$. The trees in $\widehat{\mathcal{T}}_r \setminus \{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$ will be used to write expansions for the right-hand side of (4.1.1). We remark that non-leaf nodes in $\widehat{\mathcal{T}}_r$ have three offspring, for instance $\mathcal{I}(\Xi)^2\mathcal{I}(\mathbf{1}) \in \widehat{\mathcal{T}}_r$ but $\text{  } = \mathcal{I}(\Xi)^2 \notin \widehat{\mathcal{T}}_r$. However, the three different permutations of $\mathcal{I}(\Xi)^2\mathcal{I}(\mathbf{1})$ will play the role of that $\text{}$ did in expressions like (4.2.4), and as an example of how this simplifies our combinatorics we remark that this allows us to forget about the “3” that appears in (4.2.4).

We also define a corresponding set of planted trees $\widehat{\mathcal{T}}_l = \{\mathcal{I}(\tau) : \tau \in \widehat{\mathcal{T}}_r\}$. The planted

trees in $\widehat{\mathcal{T}}_l$ will be used to describe an expansion of the solution ϕ to (4.1.1).

At certain points of our argument the roles of the planted trees of $\widehat{\mathcal{T}}_l$ and the unplanted trees of $\widehat{\mathcal{T}}_r$ will be quite different. For this reason we will reserve the use of the greek letter τ (and $\overline{\tau}$, $\widetilde{\tau}$, etc.) for elements of $\widehat{\mathcal{T}}_r$. If we want to refer to a tree that could belong to either $\widehat{\mathcal{T}}_l$ or $\widehat{\mathcal{T}}_r$ we will use the greek letter σ.

4.3.1 The order of a tree and truncation

We give a recursive definition of the order $|\cdot|$ on $\widehat{\mathcal{T}}_r \cup \widehat{\mathcal{T}}_l$ as follows. Given $\mathcal{I}(\tau) \in \widehat{\mathcal{T}}_l$ we set $|\mathcal{I}(\tau)| = |\tau| + 2$. Given $\tau \in \widehat{\mathcal{T}}_r$ we set

$$|\tau| := \begin{cases} -2, & \tau = \mathbf{1} \ , \\ -1, & \tau = \mathbf{X}_i \ , i \in \{1, \dots, d\} \\ -3 + \delta, & \tau = \Xi \ , \\ \sum_{i=1}^3 |\mathcal{I}(\tau_i)| = 6 + \sum_{i=1}^3 |\tau_i|, & \tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \ . \end{cases}$$

The values of -2 and -1 for homogeneities of the trees $\mathbf{1}$ and $\mathbf{X}_i$ may seem a bit odd but this is just due to the convention that it is $\mathcal{I}(\mathbf{1})$ and $\mathcal{I}(\mathbf{X}_i)$ that actually play the role of the classical monomials and we want $|\mathcal{I}(\mathbf{1})| = 0$ and $|\mathcal{I}(\mathbf{X}_i)| = 1$. We find that treating the classical monomials as planted trees makes our combinatorial arguments and various inductive proofs cleaner.

We now restrict the set of trees we work with and organise them into various sets. We define the following subsets of $\widehat{\mathcal{T}}_r$:

$$\begin{aligned} \text{Poly} :=& \{\mathbf{X}_1, \dots, \mathbf{X}_d, \mathbf{1}\} \ , \\ \mathcal{W} :=& \{\tau \in \widehat{\mathcal{T}}_r : \ |\tau| < -2\} \ , \ \mathring{\mathcal{W}} := \mathcal{W} \setminus \{\Xi\}, \\ \mathcal{N} :=& \{\tau \in \widehat{\mathcal{T}}_r \ , \ -2 \leqslant |\tau| \leqslant 0\}, \ \mathring{\mathcal{N}} := \mathcal{N} \setminus \text{Poly} \ . \end{aligned}$$

As a mnemonic, $\mathring{\mathcal{W}}$ (resp $\mathring{\mathcal{N}}$), is the set of those trees in $\mathcal{W}$ (resp $\mathcal{N}$) which are themselves the tree product of three planted trees.

Assumption 4.3.2. *For the rest of the thesis, we treat $\delta > 0$ as fixed, and assume, without loss of generality for the purposes of our main theorem, that δ has been chosen so that $\{|\tau| : \tau \in \mathcal{W} \cup \mathring{\mathcal{N}}\}$ does not contain any integers.*

It will be helpful in the following to have a notation for counting the number of occurrences of a certain leaf type in a tree. We define the functions $m_{\mathbf{1}}, \ m_{\mathbf{X}_i}, \ m_{\Xi} : \widehat{\mathcal{T}}_r \to \mathbb{Z}_{\geq 0}$ which count, on any given tree, the number of occurrences of $\mathbf{1}$, $\mathbf{X}_i$ and Ξ as leaves in the tree.

We also set $m_{\mathbf{x}} = \sum_{i=1}^{d} m_{\mathbf{x}_i}$ for the function that returns the total number of $\{\mathbf{X}_1, \cdots, \mathbf{X}_d\}$ leaves and $m = m_{\mathbf{1}} + m_{\mathbf{x}} + m_{\Xi}$ which returns the total number of leaves of the given tree.

One can easily check that, for $\tau \in \widehat{\mathcal{T}}_r$,

$$|\tau| = -3 + m_{\Xi}(\tau)\delta + m_{\mathbf{1}}(\tau) + 2m_{\mathbf{x}}(\tau) \; . \tag{4.3.1}$$

We now describe the roles of the various sets defined earlier. As mentioned earlier, the trees of Poly will not, by themselves, play a role in our expansions. The set $\mathcal{W}$ consists of those trees that appear in our expansion that have the lowest orders. When "subtracting the most irregular terms" as described in Section 4.2.3 we will be subtracting the trees of $\mathcal{I}(\mathcal{W})$ which are all of negative order themselves. In particular, the trees of $\mathcal{W}$ will appear in tree expansions for the right-hand side of (4.1.1) but will *not* appear by themselves on the right-hand side of the remainder equation.

On the other hand, the trees of $\mathring{\mathcal{N}}$ will appear on the right-hand side of expansions of both (4.1.1) and the remainder equation. We do not include $|\tau| > 0$ in $\tau \in \mathring{\mathcal{N}}$ since we only need to expand the right-hand side of the remainder equation up to order 0.

Our remainder will then be described by an expansion in terms of trees of $\mathcal{N}$ where the trees in $\mathcal{I}(\mathrm{Poly})$ will come with "generalised derivatives".

We have the following straightforward lemma.

Lemma 4.3.3. *The sets $\mathcal{W}$ and $\mathcal{N}$ are both finite.*

Proof. From the formula (4.3.1), one can see that $\tau \in \mathcal{W}$ if and only if $m_{\mathbf{1}}(\tau) = m_{\mathbf{x}}(\tau) = 0$ and $m_{\Xi}(\tau) < \delta^{-1}$. Similarly for $\tau \in \mathring{\mathcal{N}}$, one has

$$(m_{\mathbf{1}}(\tau), m_{\mathbf{x}}(\tau)) \in \{(0,0), (1,0), (2,0), (0,1)\} \; ,$$

and $m_{\Xi}(\tau) < \delta^{-1}(3 - m_{\mathbf{1}}(\tau) - 2m_{\mathbf{x}}(\tau))$. □

Remark 4.3.4. Clearly Lemma 4.3.3 would be false for $\delta = 0$, that is when the equation is critical.

We also have the following lemma describing the trees in $\mathring{\mathcal{W}}$.

Lemma 4.3.5. *For any $\tau \in \widehat{\mathcal{T}}_r \setminus \{\Xi\}$, $|\tau| \geq -3 + 3\delta > |\Xi| = -3 + \delta$. Moreover, for any $w \in \mathring{\mathcal{W}}$, one has*

$$w = \mathcal{I}(w_1)\mathcal{I}(w_2)\mathcal{I}(w_3) \tag{4.3.2}$$

where $w_1, w_2, w_3 \in \mathcal{W}$.

Proof. The first statement about $|\tau|$ is a simple consequence of (4.3.1) and the constraint that $m_{\Xi}(\tau) \geq 3$. For the second statement, we write $w = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$ with $\tau_1, \tau_2, \tau_3 \in \widehat{\mathcal{T}}_r$. Then we have, by bounding the orders of τ_2 and τ_3 from below,

$$|w| = 6 + |\tau_1| + |\tau_2| + |\tau_3| \geq |\tau_1| + 2\delta$$

Then, the condition that $|w| \leq -2$ forces $|\tau_1| < -2$ so we have $\tau_1 \in \mathcal{W}$ and clearly the same argument applies for τ_2, τ_3. □

We also define

$$\mathcal{T}_r := \mathcal{W} \cup \mathring{\mathcal{N}},\ \mathcal{T}_l := \mathcal{I}(\mathcal{T}_r) \cup \mathcal{I}(\mathrm{Poly}),\ \text{and } \mathcal{T} := \mathcal{T}_r \cup \mathcal{T}_l\ . \tag{4.3.3}$$

Above, and in what follows, given $A \subset \widehat{\mathcal{T}}_r$ we write $\mathcal{I}(A) = \{\mathcal{I}(\tau) :\ \tau \in A\}$.

Our various tree expansions will be linear combinations of trees in $\mathcal{T}$ and we define tree products of such linear combinations by using linearity. However, here we implement a truncation convention that will be in place for the rest of the paper. Namely, given $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{T}_l$ we enforce that if $|\sigma_1| + |\sigma_2| + |\sigma_3| > 0$, then

$$\sigma_1\sigma_2\sigma_3 := 0\ .$$

In particular, with these conventions an important identity for us will be

$$\Big(\sum_{\tau \in \mathcal{N} \cup \mathcal{W}} \mathcal{I}(\tau)\Big)^3 = \sum_{\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} \tau\ . \tag{4.3.4}$$

Above, on the left, the $(\bullet)^3$ indicates a three-fold tree product.

4.3.2 Local products

In this section we begin to specify how trees are mapped into analytic expressions. Our starting point for this will be what we call a *local product* and will be denoted by $\mathbb{X}$. Each local product $\mathbb{X}$ should be thought of as a (minimal) description of how products of planted trees should be interpreted at a concrete level.

We will view local products as being defined on a relatively small set of trees and then canonically extended to all of $\mathcal{T}$ (and in the sequel, to larger sets of trees that will appear).

Definition 4.3.6. *We define $\mathcal{Q} \subset \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$ to consist of all trees $\tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$ satisfying the following properties:*

- $\tau_1, \tau_2, \tau_3 \notin \{\mathbf{X}_1, \dots, \mathbf{X}_d\}$.

- *At most one of the trees τ_1, τ_2, τ_3 is equal to* $\mathbf{1}$.

Note that $\mathring{\mathcal{W}} \subset \mathcal{Q}$. The set $\mathcal{Q}$ includes all "non-trivial" products of trees, namely those corresponding to classically ill-defined products of distributions. Our philosophy is that once a local product $\mathbb{X}$ is specified on the noise Ξ and all these non-trivial products then we are able to define all other products that appear in our analysis.

We impose the first of the two constraints stated above because multiplication by the tree $\mathcal{I}(\mathbf{X}_i)$ corresponds to multiplication of a distribution/function by z_i which always well-defined - this makes it natural to enforce that this product is not deformed. We impose the second of the two constraints above since a tree of the form $\mathcal{I}(\mathbf{1})\mathcal{I}(\tau)\mathcal{I}(\mathbf{1})$ (or some permutation thereof) doesn't really represent a new non-trivial product because the factors $\mathcal{I}(\mathbf{1})$ corresponds to the the classical monomial 1.

Definition 4.3.7. *A* local product *is a map $\mathbb{X} : \mathcal{Q} \cup \{\Xi\} \to C^\infty(\mathbb{R} \times \mathbb{R}^d)$, which we write $\tau \mapsto \mathbb{X}_\bullet \tau$.*

We further enforce that if $\tau, \overline{\tau} \in \mathcal{Q}$ differ from each other only due to the non-commutativity of the tree-product then $\mathbb{X}_\bullet \tau = \mathbb{X}_\bullet \overline{\tau}$, that is $\mathbb{X}$ must be insensitive to the non-commutativity of the tree product.

4.3.3 Extension of local products

We now describe how any local product $\mathbb{X}$ is extended to $\mathcal{T}$, this procedure will involve induction in $m_e(\sigma) + m_{\mathbf{x}}(\sigma)$ where $m_e(\sigma)$ is the number of edges of σ.

We start by defining, for any function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(\mathcal{L}^{-1} f)$ to be the unique bounded solution u of

$$(\partial_t - \Delta)u = \rho f. \tag{4.3.5}$$

where ρ is a smooth cutoff function with value 1 in a neighbourhood of D and vanishes outside of $\{z;\ d(z,0) < 2\}$.

We now describe how we extend $\mathbb{X}$ to $\mathring{\mathcal{N}} \setminus \mathcal{Q}$. If $\tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \mathring{\mathcal{N}} \setminus \mathcal{Q}$ then precisely one of the following conditions holds

1. Exactly one of the τ_1, τ_2, τ_3 belong to the set $\{\mathbf{X}_i\}_{i=1}^d$.
2. Two of the factors τ_1, τ_2, τ_3 are equal to $\mathbf{1}$.

In the first case above we can assume without loss of generality that $\tau_1 = \mathbf{X}_i$, then we set

$$\mathbb{X}_z \tau = z_i \mathbb{X}_z(\mathcal{I}(\mathbf{1})\mathcal{I}(\tau_2)\mathcal{I}(\tau_3))\ .$$

In the second case above we can assume without loss of generality that $\tau_1 = \tau_2 = \mathbf{1}$,

then we set

$$\mathbb{X}_z\tau = (\mathcal{L}^{-1}\mathbb{X}_\bullet\tau_3)(z)\ .$$

Next we extend any local product $\mathbb{X}$ to $\mathcal{T}_l$ by setting, for any $\mathcal{I}(\tau) \in \mathcal{T}_l$,

$$\mathbb{X}_z\mathcal{I}(\tau) = \begin{cases} z_i & \text{if } \tau = \mathbf{X}_i \\ 1 & \text{if } \tau = \mathbf{1} \\ (\mathcal{L}^{-1}\mathbb{X}_\bullet\tau)(z) & \text{otherwise .} \end{cases} \tag{4.3.6}$$

Finally, we extend by linearity to allow $\mathbb{X}$ to act on linear combinations of elements of $\mathcal{T}$. Adopting the language of rough path theory and regularity structures, given smooth noise $\xi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ we say a local product $\mathbb{X}$ is a *lift* of ξ if $\mathbb{X}_z\Xi = \xi(z)$. Without additional constraints lifts are not unique.

Definition 4.3.8. *We say a local product $\mathbb{X}$ is* multiplicative *if, for every*

$$\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \mathcal{Q}\ ,$$

one has

$$\mathbb{X}_z\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) = \mathbb{X}_z\mathcal{I}(\tau_1)\mathbb{X}_z\mathcal{I}(\tau_2)\mathbb{X}_z\mathcal{I}(\tau_3)\ , \tag{4.3.7}$$

where on the right-hand side we are using the extension of $\mathbb{X}$ to planted trees.

The following lemma is then straightforward to prove.

Lemma 4.3.9. *Given any smooth $\xi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ there is a unique multiplicative lift of ξ into a local product, up to the choice of cutoff function ρ.*

Multiplicative local products will not play a special role in our analysis but we will use them at several points to compare our solution theory for (4.1.1) to the classical solution theory.

4.4 The coproduct

As discussed in Section 4.2.4, local products $\mathbb{X}_\bullet$ enter in our analysis in a centred form which depends on the choice of a basepoint x. The construction of these centred objects is given in Section 4.5. As a preliminary step, in this section we define a combinatorial operation on trees, called the coproduct, which plays a central role in this construction.

4.4.1 Derivative edges

For trees σ with $|\sigma| \in (1,2)$ we will need the centring procedure to generate first order Taylor expansions in spatial directions. In order to encode these derivatives at the level of our algebraic symbols we introduce a new set of edges $\mathcal{I}_i^+, i = 1...d$ and also define the sets of trees

$$\widetilde{\mathcal{N}} = \{\tau \in \mathring{\mathcal{N}} : -1 < |\tau| < 0\} ,$$
$$\mathcal{T}_+ = \mathcal{T}_r \cup \mathcal{T}_l \cup \left\{\mathcal{I}_i^+(\tau) : 1 \le i \le d,\ \tau \in \widetilde{\mathcal{N}} \cup \{\mathbf{X}_i\}\right\} .$$

Given $1 \le i \le d$ and $\tau \in \widetilde{\mathcal{N}} \cup \{\mathbf{X}_i\}$, we also call $\mathcal{I}_i^+(\tau)$ a planted tree. The order of these new planted trees introduced here is given by $|\mathcal{I}_i^+(\tau)| = |\tau| + 1$. We also adopt the shorthand that, for $1 \le i \le d$,

$$\mathcal{I}_i^+(\mathbf{X}_j) = 0 \text{ for } j \neq i \text{ and } \mathcal{I}_i^+(\tau) = 0 \text{ for all } \tau \in \mathring{\mathcal{N}} \setminus \widetilde{\mathcal{N}} .$$

We emphasise that these new edges will only ever appear as the bottom edge of a planted tree. Graphically we distinguish these edges by writing an index by them. For example,

$$\mathcal{I}_i^+(\mathbf{X}_j) = \overset{j}{\bullet}\!\!\mid_i = 0 \text{ if } i \neq j, \quad \mathcal{I}_j^+(\mathcal{I}(\Xi)\mathcal{I}(\Xi)\mathcal{I}(\mathbf{X}_i)) = \overset{i}{\bullet}\!\!\curlyvee_j .$$

At an analytic level, the role of $\mathcal{I}_i^+(\mathbf{X}_j)$ is the same as that of $\mathcal{I}(\mathbf{1})$ but distinguishing these symbols will be important - see Remark 4.4.3.

4.4.2 Algebras and vector spaces of trees

We now give some notation for describing the codomain of our coproduct Δ. Given a set of trees T we write $\mathrm{Vec}(T)$ for the vector space (over $\mathbb{R}$) generated by T.

Given a set of planted trees T we write $\mathrm{Alg}(T)$ for the unital non-commutative algebra (again over $\mathbb{R}$) generated by T. We will distinguish between the tree product introduced in Section 4.3 and the product that makes $\mathrm{Alg}(T)$ an algebra, calling the the latter product the "forest product". While both the tree product and forest product are non-commutative, the roles they play are quite different - see Remark 4.4.2

We will write $\cdot$ to denote the forest product when using algebraic variables for trees, that is given $\sigma, \widetilde{\sigma} \in T$, we write $\sigma \cdot \widetilde{\sigma}$ for the forest product of σ and $\widetilde{\sigma}$. As a real vector space $\mathrm{Alg}(T)$ is spanned by products $\sigma_1 \cdot \sigma_2 \cdots \sigma_n$ with $\sigma_1, \dots, \sigma_n \in T$. We call such a product $\sigma_1 \cdot \sigma_2 \cdots \sigma_n$ a "forest". The unit for the forest product is given by the "empty" forest and is denoted by 1.

Graphically, we will represent forest products just by drawing the corresponding planted

trees side by side, for instance writing

.

4.4.3 Coproduct

We define another set of trees

$$\mathcal{T}^{\mathrm{rec}} := \mathcal{I}(\mathcal{N}) \cup \{\mathcal{I}_i^+(\tau) : \tau \in \widetilde{\mathcal{N}} \cup \{\mathbf{X}_i\}, 1 \le i \le d\} \ . \tag{4.4.1}$$

Our coproduct will be a map

$$\Delta : \mathcal{T}_+ \to \mathrm{Vec}(\mathcal{T}_+) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}}) \ .$$

Our definition of Δ will be recursive. The base cases of this recursive definition are the trees in $w \in \mathcal{W}$ and planted trees $\mathcal{I}(w),\ w \in \mathcal{W}$, and the elementary trees $\mathcal{I}(\mathbf{1}), \mathcal{I}(\mathbf{X}_i)$ and $\mathcal{I}_i^+(\mathbf{X}_i)$:

$$\begin{aligned}
&\Delta\mathcal{I}(\mathbf{1}) = \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{1}), \\
&\Delta\mathcal{I}(\mathbf{X}_i) = \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{X}_i) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\mathbf{X}_i), \\
&\Delta\mathcal{I}_i^+(\mathbf{X}_i) = \mathcal{I}_i^+(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\mathbf{X}_i), \\
&\Delta w = w \otimes 1,\ \Delta\mathcal{I}(w) = \mathcal{I}(w) \otimes 1,\ w \in \mathcal{W}.
\end{aligned} \tag{4.4.2}$$

Note that in the last line, the 1 appearing is the unit element in the algebra and should not be mistaken for $\mathbf{1} \in \mathrm{Poly}$.

The recursive part of our definition is then given by

$$\begin{aligned}
&\Delta\mathcal{I}(\tau) = \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) + (\mathcal{I} \otimes \mathrm{Id})\Delta\tau,\ \tau \in \mathring{\mathcal{N}}, \\
&\Delta\mathcal{I}_i^+(\tau) = \mathcal{I}_i^+(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) + (\mathcal{I}_i^+ \otimes \mathrm{Id})\Delta\tau, \tau \in \widetilde{\mathcal{N}}, \\
&\Delta(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)) = \Delta(\mathcal{I}(\tau_1))\Delta(\mathcal{I}(\tau_2))\Delta(\mathcal{I}(\tau_2)),\ \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \mathring{\mathcal{N}},
\end{aligned} \tag{4.4.3}$$

where, on the right-hand side of the last line above we are referring to the natural product $[\mathrm{Vec}(\mathcal{T}_l) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})]^{\otimes 3} \to \mathrm{Vec}(\mathcal{T}_r) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$. This product is just given by setting

$$\otimes_{i=1}^3 (\mathcal{I}(\tau_i) \otimes a_i) \mapsto \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \otimes (a_1 \cdot a_2 \cdot a_3) \ , \tag{4.4.4}$$

and then extending by linearity. We also make note of the fact that, in the first and second lines of (4.4.3), we are using our convention of Einstein summation - since i does not appear on the left-hand side then the i on the right-hand side is summed from 1 to d.

One can verify that it is indeed the case that Δ maps $\mathcal{T}_+$ into $\mathrm{Vec}(\mathcal{T}_+) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$ by checking inductively, using (4.4.2) for the bases cases and (4.4.3) for the inductive step. We also have the following lemma on how Δ acts on subsets of $\mathcal{T}_+$.

Lemma 4.4.1. *We have that*

- Δ *maps* $\mathcal{T}_l$ *into* $\mathrm{Vec}(\mathcal{T}_l) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$,
- Δ *maps* $\mathcal{T}_r$ *into* $\mathrm{Vec}(\mathcal{T}_r) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$,
- Δ *maps* $\mathring{\mathcal{N}}$ *into* $\mathrm{Vec}(\mathring{\mathcal{N}}) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$, *and*
- Δ *maps* $\mathcal{T}^{\mathrm{rec}}$ *into* $\mathrm{Vec}(\mathcal{T}^{\mathrm{rec}}) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$.

Proof. The first two statements are immediate consequences of our definitions. We turn to proving the third statement, where we proceed by induction in the number of edges of $\tau \in \mathcal{N}$. The base case(s) where τ has three edges are easily verified by hand. For the inductive step, we write $\tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$. Now, if $\tau_1, \tau_2, \tau_3 \in \mathcal{W}$ one can check that the last line of (4.4.3) gives us that $\Delta\tau = \tau \otimes 1$ and we are done. On the other hand, if we have $\tau_i \in \mathcal{N}$ for some i then $\Delta\mathcal{I}(\tau) \in \mathrm{Vec}(\mathcal{I}(\mathcal{N})) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$ and so we are done by combining the last line of (4.4.3) with (4.3.2).

Finally, the fourth statement is immediate by inspection for planted trees $\mathcal{T}^{\mathrm{rec}} \setminus \mathcal{I}(\mathring{\mathcal{N}})$ while for planted trees in $\mathcal{I}(\mathring{\mathcal{N}})$ it follows from using the third statement for τ in the first line of (4.4.3). □

We extend Δ to sums of trees by linearity, so that $\Delta : \mathrm{Vec}(\mathcal{T}_+) \to \mathrm{Vec}(\mathcal{T}_+) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$.

Remark 4.4.2. The two products we have introduced on trees, the tree product and the forest product, play different roles in our framework: the tree product represents a point-wise product of functions/distributions which may not be defined canonically. Therefore we do *not* enforce that local products act multiplicatively with respect to the tree product. On the other hand, any map on trees that is applied to a forest is extended multiplicatively - we do not allow for any flexibility in how forest products are interpreted at a concrete level. In particular, the trees in the forests that Δ produces in the right factor of its codomain are all of non-negative order and should be thought of as being associated to products of base-point dependent constants rather than a point-wise product of space-time functions/distributions.

Remark 4.4.3. The coproduct plays a central role in algebraically encoding the terms that appear in our centring procedure, but the precise choices (4.4.2) and (4.4.3) are also motivated by additional properties we will need from the coproduct, namely the self-commutativity property (4.4.5) and the combinatorial identity (4.4.7) .

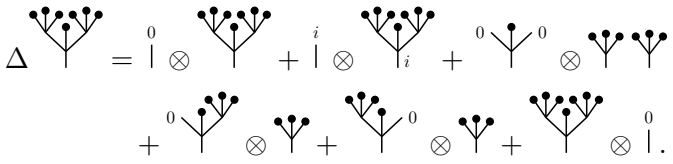
The key content of (4.4.7) is that for any $\tau \in \mathcal{N}$ and any $\sigma \otimes \sigma_1 \cdots \sigma_n$ appearing in the expansion of $\Delta\mathcal{I}(\tau)$, the precise number of $\mathbf{X}_i$ and $\mathbf{1}$ generators appearing in the forest

$\sigma_1 \cdots \sigma_n$ and in τ coincide. This in turn is needed for the crucial relation (4.6.20). For this reason we have to work with the different algebraic objects $\mathcal{I}(\mathbf{1})$, $\mathcal{I}_i^+(\mathbf{X}_i)$, and the empty forest 1 - even though all these symbols are treated identically at an analytic level.

For instance, one might be tempted to set $\Delta\mathcal{I}(\mathbf{1}) = \mathcal{I}(\mathbf{1}) \otimes 1$ but this would break (4.4.7) since the number of $\mathbf{1}$'s in the empty forest 1 is zero while the number in $\mathbf{1}$ is one. Similarly, one cannot include $\mathcal{I}(\mathbf{X}_i) \otimes \mathbf{1}$ or $\mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}(\mathbf{1})$ in the expansion of $\Delta\mathcal{I}(\mathbf{X}_i)$. Finally, we have to write the term $\mathcal{I}_i^+(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau)$ in the second line of (4.4.3) instead of say, $\mathcal{I}(\mathbf{1}) \otimes \mathcal{I}_i^+(\tau)$, because of our earlier choices and (4.4.5).

Example 4.4.4. We show one pictorial example. The value of our parameter δ influences the definition of the set $\mathring{\mathcal{N}}$ and whether $\mathcal{I}_i^+(\tau)$ vanishes or not for $\tau \in \mathring{\mathcal{N}}$, therefore most non-trivial computations of Δ we would present are valid only for a certain range of the parameter δ.

For the example we present below, we restrict to $\frac{3}{7} > \delta > \frac{1}{3}$ and therefore $1 > |$  $| > 0$ and $2 >$  $= -1 + 7\delta > 1$. We then have, using Einstein's convention for the index $i \in \{1, ..., d\}$ (when an index i appears twice on one side of an equation, it means a summation over $i = 1...d$)



We show now the example of an unplanted tree in the case $\delta < \frac{1}{3}$ and therefore $|$  $| < 0$. On the other hand, we always have $|$  $| = 1 + 2\delta > 1$. With Einstein's convention for the index $j \in \{1, ..., d\}$

Remark 4.4.5. The last formula in (4.4.3) for $\tau \in \mathring{\mathcal{N}}$ is also valid for $\tau \in \mathring{\mathcal{W}}$ where it is trivial. The first formula of (4.4.3) can also be extended to $\tau \in \mathcal{N}$ if one adopts the convention that $\Delta\mathbf{X}_i = \Delta\mathbf{1} = 0$. We chose not to do this since the trees of Poly do not, by themselves, play a role in our algebraic expansions and analysis except when they appear in a larger tree.

We extend Δ to forests of planted trees by setting $\Delta 1 = 1 \otimes 1$ and, for any forest $\sigma_1 \cdots \sigma_n$, $n \geq 1$,

$$\Delta(\sigma_1 \cdots \sigma_n) = (\Delta\sigma_1) \cdots (\Delta\sigma_n) \ ,$$

where on the right-hand side we use the forest product to multiply all the factors components-wise. We extend to sums of forests of planted trees by additivity so that $\Delta : \mathrm{Alg}(\mathcal{T}_l) \to \mathrm{Alg}(\mathcal{T}_l) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$ and, by Lemma 4.4.1, we also have that Δ maps $\mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$ into $\mathrm{Alg}(\mathcal{T}^{\mathrm{rec}}) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$.

While a single application of Δ will be used for centring objects around a basepoint, we will see in Section 4.5 that a double application of Δ will be used for describing the behaviour when changing this basepoint. This is our reason for also defining Δ on the planted trees of $\mathcal{T}^{\mathrm{rec}} \setminus \mathcal{T}_l$. It will be important below to know that the two ways of "applying Δ twice" agree, this is encoded in the following lemma.

Lemma 4.4.6. *Δ satisfies a co-associativity property: for any $\sigma \in \mathcal{T}$ one has*

$$(\Delta \otimes \mathrm{Id})\Delta\sigma = (\mathrm{Id} \otimes \Delta)\Delta\sigma \,, \tag{4.4.5}$$

where both sides are seen as elements of $\mathrm{Vec}(\mathcal{T}) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}}) \otimes \mathrm{Alg}(\mathcal{T}^{\mathrm{rec}})$.

Proof. We argue by induction in the size of σ. The cases where $\sigma = \mathcal{I}(\mathbf{1})$, or $\mathcal{I}(\mathbf{X}_i)$ are straightforward to check. Note that by multiplicativity of Δ with respect to the tree product it suffices to establish the inductive step for $\sigma = \mathcal{I}(\tau)$ for some $\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$. The case where $\tau \in \mathring{\mathcal{W}}$ is trivial so we walk through the verification of the identity when $\tau \in \mathring{\mathcal{N}}$. On the left we have

$$\begin{aligned}
(\Delta \otimes \mathrm{Id})\Delta\mathcal{I}(\tau) =& (\Delta \otimes \mathrm{Id})\Big[\mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) + (\mathcal{I} \otimes \mathrm{Id})\Delta\tau\Big] \\
=& \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) \\
&+ \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) + \mathcal{I}(\mathbf{1}) \otimes \big((\mathcal{I} \otimes \mathrm{Id})\Delta\tau\big) \\
&+ \mathcal{I}(\mathbf{X}_j) \otimes \big((\mathcal{I}_j^+ \otimes \mathrm{Id})\Delta\tau\big) + (\mathcal{I} \otimes \mathrm{Id} \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id})\Delta\tau \,.
\end{aligned}$$

On the right we have

$$\begin{aligned}
(\mathrm{Id} \otimes \Delta)\Delta\mathcal{I}(\tau) =& (\mathrm{Id} \otimes \Delta)\Big[\mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) + (\mathcal{I} \otimes \mathrm{Id})\Delta\tau\Big] \\
=& \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau) + \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{X}_j) \otimes \mathcal{I}_j^+(\tau) \\
&+ \mathcal{I}(\mathbf{1}) \otimes \big((\mathcal{I} \otimes \mathrm{Id})\Delta\tau\big) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) \\
&+ \mathcal{I}(\mathbf{X}_i) \otimes \big((\mathcal{I}_i^+ \otimes \mathrm{Id})\Delta\tau\big) + (\mathcal{I} \otimes \mathrm{Id} \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta)\Delta\tau \,.
\end{aligned}$$

All the terms in the expression for the left- and right-hand sides can be immediately matched except for the very last terms, but these are seen to be identical by using our

induction hypothesis which is

$$(\mathrm{Id} \otimes \Delta)\Delta\tau = (\Delta \otimes \mathrm{Id})\Delta\tau \ .$$

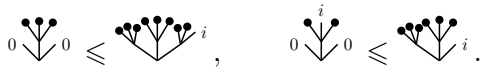
□

4.4.4 Useful relations on trees

We introduce two (reflexive and antisymmetric) relations on unplanted trees $\mathcal{T}_r$, which we denote $\leqslant$ and $\subset$.

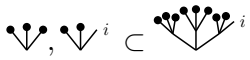
Definition 4.4.7. *Given $\overline{\tau}, \tau \in \mathcal{T}_r$ we have $\overline{\tau} \leqslant \tau$ if and only if one can obtain τ from $\overline{\tau}$ by replacing occurrences of the generators $\mathbf{1}$ in $\overline{\tau}$ with appropriately chosen trees $\tau_1, ...\tau_{m_{\mathbf{1}}(\overline{\tau})} \in \mathcal{T}_r$ and, for every $1 \leq i \leq d$, occurrences of $\mathbf{X}_i$ with trees $\tau_1, \dots, \tau_{m_{\mathbf{X}_i(\overline{\tau})}} \in \mathcal{T}_r \setminus \{\mathbf{1}, \mathbf{X}_j, j \neq i\}$.*

Example 4.4.8. We give two pictorial examples



Definition 4.4.9. *Given $\overline{\tau}, \tau \in \mathcal{T}_r$ we have $\overline{\tau} \subset \tau$ if and only if $\overline{\tau} = \tau$ or $\overline{\tau}$ appears in the inductive definition of τ, that is the expression $\mathcal{I}(\overline{\tau})$ should appear at some point when one writes out the full algebraic expression for τ.*

Example 4.4.10. We give an example below.



We also use the notation $<$ and $\subsetneq$ to refer to the non-reflexive (strict) relations corresponding to $\leq$ and $\subset$.

One can get an intuition of how the coproduct works with the idea of cutting branches: on the left-hand side of $\Delta\tau$ we have trees $\overline{\tau} \leqslant \tau$, and on the right-hand side, we have the trees $\mathcal{I}(\widetilde{\tau})$ or $\mathcal{I}_i^+(\widetilde{\tau})$ where $\widetilde{\tau} \subset \tau$ has been cut from τ to obtain $\overline{\tau}$. We formalise this in in the following section.

4.4.5 Another formula for Δ

We write $\mathcal{F}^{\mathrm{rec}}$ for the collection of all finite, non-commutative words in $\mathcal{T}^{\mathrm{rec}}$, including the empty word. In particular, $\mathcal{F}^{\mathrm{rec}}$ is a vector space basis for $\mathrm{Alg}(\mathcal{T}_+)$. We define a map $C_+ : \mathcal{T}_r \times \mathcal{T}_r \to \mathcal{F}^{\mathrm{rec}}$ recursively. The recursion is given in the following table:

Here p_+ is the projection on trees of positive order. In particular, for $\delta < 1$, one has

$\tau \setminus \overline{\tau}$	$\mathbf{1}$	$\mathbf{X}_i$	Ξ	$\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(\overline{\tau}_3)$
$\mathbf{1}$	$\mathcal{I}(\mathbf{1})$	0	0	0
$\mathbf{X}_j$	$\mathcal{I}(\mathbf{X}_j)$	$\mathcal{I}_i^+(\mathbf{X}_j)$	0	0
Ξ	0	0	1	0
$\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$	$p_+\mathcal{I}(\tau)$	$\mathcal{I}_i^+(\tau)$	0	$C_+(\overline{\tau}_1,\tau_1)\cdot C_+(\overline{\tau}_2,\tau_2)\cdot C_+(\overline{\tau}_3,\tau_3)$

Table 4.1: This table gives a recursive definition of $C_+(\overline{\tau},\tau)$. Possible values of τ are displayed in the first column, while possible values of $\overline{\tau}$ are shown in the first row. The corresponding values of $C_+(\overline{\tau},\tau)$ are shown in the remaining fields.

$p_+\mathcal{I}(\Xi)$, $\mathcal{I}_i^+(\Xi) = 0$. Note also that $C_+(\Xi,\Xi) = 1$, which is the unit element in the algebra of trees, not to be mistaken for $\mathbf{1}$.

Example 4.4.11. We give two pictorial examples

$$C_+(\ {}^{0}\!\!\bigvee\!\!{}^{0}\ ,\ \bigvee^{i}\ ) = \begin{cases} \bigvee\!\bigvee^{i} & \text{if } \delta > \frac{1}{3} \\ 0 & \text{else.} \end{cases}$$

and

$$C_+(\ \bigvee^{j}\ ,\ \bigvee^{i}\ ) = \bigvee_{j}^{i}\ .$$

We explain how this can be understood in the language of "cuts". There are three types of cutting procedures that can be applied to a tree σ.

1. One cuts an $\mathcal{I}$-branch and takes the attached planted tree, leaving behind an $\mathcal{I}(\mathbf{1})$.
2. One cuts an $\mathcal{I}$ branch and takes the attached planted tree, with its "trunk" becoming a derivative $\mathcal{I}_k^+$, and leaving behind a $\mathcal{I}(\mathbf{X}_k)$. Note that this only occurs when one the tree $\tau \subset \sigma$ attached to this $\mathcal{I}$ branch belongs to $\widetilde{\mathcal{N}}$.
3. One cuts an $\mathcal{I}_i^+$ branch (which must be the trunk of σ) and takes the whole tree σ, leaving behind an $\mathcal{I}_i^+(\mathbf{X}_i)$.

If $\delta > \frac{1}{3}$, the tree ${}^{0}\!\!\bigvee\!\!{}^{0}$ is obtained from $\bigvee^{i}$ by performing the first type of cut on the leftmost and rightmost $\mathcal{I}$ branches of $\bigvee^{i}$ connected to the root, leaving behind ${}^{0}\!\!\bigvee\!\!{}^{0}$.

In the second example, one performs the second type of cut on the rightmost $\mathcal{I}$ branch connected to the root of $\bigvee^{i}$, generating an $\mathcal{I}_j^+$ trunk on the planted tree taken and leaving behind an $\mathcal{I}(\mathbf{X}_j)$ on $\bigvee^{j}$.

Some immediate properties of these forests $C_+(\tau,\overline{\tau})$ are given in the following lemma.

Lemma 4.4.12. *Let $\tau, \overline{\tau} \in \mathcal{T}_r$. Then we have*

$$C_+(\overline{\tau}, \tau) \neq 0 \Rightarrow \overline{\tau} \leqslant \tau, \tag{4.4.6}$$
$$\mathcal{I}(\tau') \in C_+(\overline{\tau}, \tau) \text{ or } \mathcal{I}_i^+(\tau') \in C_+(\overline{\tau}, \tau) \Rightarrow \tau' \subset \tau.$$

Under the assumption that $C_+(\overline{\tau}, \tau) \neq 0$ we have

$$m_\Xi(\tau) = m_\Xi(\overline{\tau}) + m_\Xi(C_+(\overline{\tau}, \tau)) \quad m_{\mathbf{1}}(\overline{\tau}) + m_{\mathbf{X}}(\overline{\tau}) = \sharp(C_+(\overline{\tau}, \tau)), \tag{4.4.7}$$
$$m_{\mathbf{1}}(\tau) = m_{\mathbf{1}}(C_+(\overline{\tau}, \tau)), \quad m_{\mathbf{X}}(\tau) = m_{\mathbf{X}}(C_+(\overline{\tau}, \tau)),$$

where $\sharp(C_+(\overline{\tau}, \tau))$ is the number of trees in $C_+(\overline{\tau}, \tau)$ (including multiplicity) and we extend the functions $m_{\mathbf{1}}$, $m_{\mathbf{X}}$ and m_Ξ to forests of the planted trees by summing over the individual planted trees in the forest.

The following lemma finally gives the expression of the coproduct Δ in terms of C_+. This expression is used throughout the chapter without explicit reference to this lemma.

Lemma 4.4.13. *For any $\tau \in \mathcal{N} \cup \mathcal{W}$,*

$$\Delta\mathcal{I}(\tau) = \sum_{\overline{\tau} \in \mathcal{N} \cup \mathcal{W}} \mathcal{I}(\overline{\tau}) \otimes C_+(\overline{\tau}, \tau). \tag{4.4.8}$$

In particular,

$$\Delta\mathcal{I}(\tau) = \begin{cases} \mathcal{I}(\tau) \otimes 1 & \text{if } \tau \in \mathcal{W}\,, \\ \sum_{\overline{\tau} \in \mathcal{N}} \mathcal{I}(\overline{\tau}) \otimes C_+(\overline{\tau}, \tau) & \text{if } \tau \in \mathcal{N}\,. \end{cases} \tag{4.4.9}$$

Moreover, for any $\tau \in \mathcal{T}_r$,

$$\Delta\tau = \sum_{\overline{\tau} \in \mathcal{N} \cup \mathcal{W}} \overline{\tau} \otimes C_+(\overline{\tau}, \tau) = \tau \otimes 1 + \sum_{\substack{\overline{\tau} \in \mathcal{N} \\ \overline{\tau} \neq \tau}} \overline{\tau} \otimes C_+(\overline{\tau}, \tau)\,. \tag{4.4.10}$$

Proof. We prove (4.4.9) by induction, with the base cases given by $\tau \in \mathrm{Poly} \cup \mathcal{W}$ which we check now.

$$\Delta\mathcal{I}(\mathbf{1}) = \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{1}) \text{ and } C_+(\overline{\tau}, \mathbf{1}) = \mathcal{I}(\mathbf{1})\delta_{\{\overline{\tau}=\mathbf{1}\}}.$$

Since $\mathcal{I}_j^+(\mathbf{X}_i) = 0$ for $j \neq i$, we also have $\Delta\mathcal{I}(\mathbf{X}_i) = \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\mathbf{X}_i) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\mathbf{X}_i)$ and $C_+(\overline{\tau}, \mathbf{X}_i) = \mathcal{I}(\mathbf{X}_i)\delta_{\{\overline{\tau}=\mathbf{1}\}} + \mathcal{I}_i^+(\mathbf{X}_i)\delta_{\{\overline{\tau}=\mathbf{X}_i\}}$.

Finally, for $\tau = w \in \mathcal{W}$, we have to show that the sum in the right-hand side of (4.4.8) contains only one term. Indeed, for any $w' \leqslant w$, we also have $w' \in \mathcal{W}$, hence $|\mathcal{I}(w')| < 0$ and $C_+(\overline{\tau}, w) = \delta_{\{\overline{\tau}=w\}}$.

We now prove the inductive step for $\tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \mathring{\mathcal{N}}$, with the induction hypothesis $\Delta\mathcal{I}(\tau_k) = \sum_{\overline{\tau}_k} \mathcal{I}(\overline{\tau}_k) \otimes C_+(\overline{\tau}_k, \tau_k)$, for $k = 1, 2, 3$. We have that $C_+(\mathbf{1}, \tau) = p_+\mathcal{I}(\tau) = \mathcal{I}(\tau)$ since $\tau \in \mathring{\mathcal{N}}$ and $C_+(\mathbf{X}_i, \tau) = \mathcal{I}_i^+(\tau) = \mathcal{I}_i^+(\tau)\delta_{\{\tau \in \widetilde{\mathcal{N}}\}}$, and from the definition of Δ, we have

$$\begin{aligned}\Delta\mathcal{I}(\tau) =& \mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau) + \mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau) \\ &+ \sum_{\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3} \mathcal{I}(\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(\overline{\tau}_3)) \otimes C_+(\overline{\tau}_1, \tau_1)C_+(\overline{\tau}_2, \tau_2)C_+(\overline{\tau}_3, \tau_3).\end{aligned}$$

Furthermore, for any $\overline{\tau} \leqslant \tau \in \mathring{\mathcal{N}}$, we have that either $\overline{\tau} = \tau$ or $m_{\mathbf{1}}(\overline{\tau}) + m_{\mathbf{x}}(\overline{\tau}) \geqslant 1$ therefore $|\overline{\tau}| \geqslant -2$. Hence the sum can be restricted to trees $\overline{\tau} = \mathcal{I}(\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(\overline{\tau}_3)) \in \mathring{\mathcal{N}}$. This concludes the proof of (4.4.8)

Finally, we prove (4.4.10). This is immediate if $\tau = \mathcal{W}$, otherwise one has $\tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \in \mathring{\mathcal{N}}$ and one obtains the desired result by combining (4.4.9) and the multiplicativity of Δ with respect to the tree product as described in the last line of (4.4.3). □

4.5 From local products to paths

4.5.1 Definition of paths and centrings

For any choice of local product $\mathbb{X}$, we will define two corresponding families of maps, a path $(\mathbb{X}_{z,x} : \mathcal{T}_+ \to \mathbb{R};\ z, x \in \mathbb{R} \times \mathbb{R}^d)$ and a centring $(\mathbb{X}_z^{\mathrm{rec}} : \mathcal{T}^{\mathrm{rec}} \to \mathbb{R};\ z \in \mathbb{R} \times \mathbb{R}^d)$ where $\mathcal{T}^{\mathrm{rec}}$ is defined in (4.4.1).

Both the path and the centring are defined through an inductive procedure that intertwines these two families of maps.

One particular aim of our definitions will be to allow us to obtain the formula

$$\mathbb{X}_{z,x}\mathcal{I}(\tau) = (\mathbb{X}_z \otimes \mathbb{X}_x^{\mathrm{rec}})\Delta\mathcal{I}(\tau) \text{ for any } \tau \in \mathring{\mathcal{N}}\,, \tag{4.5.1}$$

where we are extending $\mathbb{X}_x^{\mathrm{rec}}$ to act on forests of planted trees by multiplicativity.

As we discussed in Section 4.2, we define

$$\mathbb{X}_{z,x}\mathcal{I}(\mathbf{1}) := 1 \text{ and } \mathbb{X}_{z,x}\mathcal{I}(\mathbf{X}_i) := z_i - x_i, \tag{4.5.2}$$

and

$$\mathbb{X}_{z,x}\tau := \mathbb{X}_z\tau, \quad \mathbb{X}_{z,x}\mathcal{I}(\tau) := \mathbb{X}_z\mathcal{I}(\tau) \text{ for any } \tau \in \mathcal{W}\,. \tag{4.5.3}$$

With our definition (4.5.3) it is immediate that (4.5.1) holds for $\tau \in \mathcal{W}$. For $\tau \in \mathring{\mathcal{N}}$ we

define

$$\mathbb{X}_{z,x}\mathcal{I}(\tau) := \mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(z) - \mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(x) - \mathbb{1}_{\tau\in\widetilde{\mathcal{N}}}(z_i - x_i)\nu_\tau^{(i)}(x)\ , \quad (4.5.4)$$

for which one must take as input the definition of $\mathbb{X}_{\bullet,x}\tau$ and

$$\nu_\tau(x) = (\nu_\tau^{(i)}(x))_{i=1}^d := \nabla(\mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau))(z)|_{z=x}\ , \quad (4.5.5)$$

where ∇ denotes the spatial gradient. For the centring we will define

$$\begin{aligned}
\mathbb{X}_x^{\mathrm{rec}}\mathcal{I}(\mathbf{1}) :=& 1, \quad \mathbb{X}_x^{\mathrm{rec}}\mathcal{I}(\mathbf{X}_i) := -x_i, \quad \mathbb{X}_x^{\mathrm{rec}}\mathcal{I}_i^+(\mathbf{X}_i) := 1,\\
\mathbb{X}_x^{\mathrm{rec}}\mathcal{I}(\tau) :=& -\mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(x) + \mathbb{1}_{\tau\in\widetilde{\mathcal{N}}}x_i\nu_\tau^{(i)}(x) \text{ for any } \tau\in\mathring{\mathcal{N}},\\
\mathbb{X}_x^{\mathrm{rec}}\mathcal{I}_i^+(\tau) :=& -\nu_\tau^{(i)}(x) \text{ for any } \tau\in\widetilde{\mathcal{N}}\ .
\end{aligned} \quad (4.5.6)$$

The formulae above are inductive, we remark that for $\tau\in\mathring{\mathcal{N}}$ one needs to be given $\mathbb{X}_{\bullet,y}\tau$ in order to define $\mathbb{X}_y^{\mathrm{rec}}\mathcal{I}(\tau)$ and, if $\tau\in\widetilde{\mathcal{N}}$, that same input is needed to define $\mathbb{X}_y^{\mathrm{rec}}\mathcal{I}_i^+(\tau)$.

Finally, to handle the tree products that appear in the remainder equation we define

$$\mathbb{X}_{z,x}\tau := (\mathbb{X}_z\otimes\mathbb{X}_x^{\mathrm{rec}})\Delta\tau \ \text{ for all } \tau\in\mathring{\mathcal{N}}\ . \quad (4.5.7)$$

Again, the formula above is an inductive definition - a sufficient condition for specifying the right-hand side above is that we already know $\mathbb{X}_{\bullet,x}$ for every $\overline{\tau}\in\mathring{\mathcal{N}}$ for $\overline{\tau}\subsetneq\tau$.

Remark 4.5.1. Note that if (4.5.7) is extended to $\tau\in\mathcal{W}$ it agrees with the definition given in (4.5.3).

Lemma 4.5.2. *If one adopts the inductive set of definitions* (4.5.2)*,* (4.5.4)*,* (4.5.3)*, and* (4.5.6)*, to determine the path on $\mathcal{T}$ and the centring on $\mathcal{T}^{\mathrm{rec}}$ then* (4.5.1) *holds for every $\tau\in\mathcal{N}\cup\mathcal{W}$.*

Proof. The fact that (4.5.1) holds for every $\tau\in\mathcal{W}\cup\mathcal{I}(\mathcal{W})\cup\mathcal{I}(\mathrm{Poly})$ is immediate. Now suppose that $\tau\in\mathring{\mathcal{N}}$, we can then rewrite (4.5.4) as

$$\mathbb{X}_{z,x}\mathcal{I}(\tau) = \mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(z) - \mathbb{1}_{\tau\in\widetilde{\mathcal{N}}}z_i\nu_\tau^{(i)}(x) + \mathbb{X}_x^{\mathrm{rec}}\mathcal{I}(\tau).$$

We also have

$$\begin{aligned}
\mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(z) =& \mathcal{L}^{-1}[(\mathbb{X}_\bullet\otimes\mathbb{X}_x^{\mathrm{rec}})\Delta\tau](z)\\
=& [(\mathcal{L}^{-1}\mathbb{X}_\bullet\cdot)(z)\otimes\mathbb{X}_x^{\mathrm{rec}}]\Delta\tau\\
=& (\mathbb{X}_z\otimes\mathbb{X}_x^{\mathrm{rec}})(\mathcal{I}\otimes\mathrm{Id})\Delta\tau\ .
\end{aligned}$$

The desired claim follows upon observing that

$$\begin{aligned}-\mathbb{1}_{\tau\in\widetilde{\mathcal{N}}} z_i \nu_\tau^{(i)}(x) =& (\mathbb{X}_z \otimes \mathbb{X}_x^{\mathrm{rec}})(\mathcal{I}(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau)) \ , \\ \text{and } \mathbb{X}_x^{\mathrm{rec}}\mathcal{I}(\tau) =& (\mathbb{X}_z \otimes \mathbb{X}_x^{\mathrm{rec}})(\mathcal{I}(\mathbf{1}) \otimes \mathcal{I}(\tau)) \ .\end{aligned}$$

□

At this point we have finished the inductive definition of the path on the trees of $\mathcal{T}$ and of the centring on the trees of $\mathcal{T}^{\mathrm{rec}}$. What is left is to define the path on the trees of $\{\mathcal{I}_i^+(\tau) : 1 \le i \le d,\ \tau \in \widetilde{\mathcal{N}} \cup \{\mathbf{X}_i\}\}$.

In keeping with our convention of thinking of $\mathcal{I}_i^+(\mathbf{X}_i)$ as acting like $\mathcal{I}(\mathbf{1})$ for all analysis, we set

$$\mathbb{X}_{z,x}\mathcal{I}_i^+(\mathbf{X}_i) := 1 \ .$$

Our definition for the action of a path $\mathbb{X}_{\bullet,\bullet}$ on a tree $\mathcal{I}_i^+(\tau)$ for $\tau \in \widetilde{\mathcal{N}}$ is motivated by the fact that such trees are not really part of our tree expansions but instead only appear in order to encode change of base-point operations.

In particular, $\mathbb{X}_{u,x}\mathcal{I}_i^+(\tau)$ will play a role in how we relate centring at u versus centring at x and the identity we will be aiming for is Chen's relation (4.5.10).

The key identity we would like to hold is that, for any $z, x \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathbb{X}_x^{\mathrm{rec}}\mathcal{I}_i^+(\tau) = \big(\mathbb{X}_z^{\mathrm{rec}} \otimes \mathbb{X}_{z,x}\big)\Delta\mathcal{I}_i^+(\tau) \ . \tag{4.5.8}$$

Note that in the above equation we are using our convention of extending $\mathbb{X}_{z,x}$ to forests of planted trees by multiplicativity.

Expanding the action of Δ in (4.5.8) gives us an inductive procedure for defining $\mathbb{X}_{\bullet,\bullet}\mathcal{I}_i^+(\tau)$ for $\tau \in \widetilde{\mathcal{N}}$. Namely, we will define, for any $\tau \in \widetilde{\mathcal{N}}$,

$$\begin{aligned}\mathbb{X}_{z,x}\mathcal{I}_i^+(\tau) :=& \mathbb{X}_x^{\mathrm{rec}}\mathcal{I}_i^+(\tau) - (\mathbb{X}_z^{\mathrm{rec}} \circ \mathcal{I}_i^+ \otimes \mathbb{X}_{z,x})\Delta\tau \\ =& -\nu_\tau^{(i)}(x) + \sum_{\overline{\tau}\in\widetilde{\mathcal{N}}} \nu_{\overline{\tau}}^{(i)}(z)\mathbb{X}_{z,x}C_+(\overline{\tau},\tau) \ .\end{aligned} \tag{4.5.9}$$

We then see that, in order to define $\mathbb{X}_{z,x}\mathcal{I}_i^+(\tau)$ it suffices to have defined $\mathbb{X}_{\bullet,x}(\tau)$, $\mathbb{X}_{\bullet,x}(\overline{\tau})$ for every $\overline{\tau} \in \mathring{\mathcal{N}}$ with $\overline{\tau} < \tau$, along with $\mathbb{X}_{z,x}\mathcal{I}(\widetilde{\tau})$ and $\mathbb{X}_{z,x}\mathcal{I}_i^+(\widetilde{\tau})$ for every $\widetilde{\tau} \subsetneq \tau$.

Remark 4.5.3. We take a moment to draw parallels between our definitions and those found in the theory of regularity structures. Those unfamiliar with the theory of regularity structures can skip this remark.

In our context, the local product $\mathbb{X}_z$ plays the role of the "un-recentred" $\mathbf{\Pi}(\bullet)(z)$ map in

the theory of regularity structures.

The corresponding path $\mathbb{X}_{z,x}$ sometimes plays the role of the map $(\Pi_x \bullet)(z)$ and sometimes plays a role more analogous to $\gamma_{z,x}(\bullet)$ where $\gamma_{z,x}$ is as in [41, Section 8.2], that is it is the character that defines $\Gamma_{z,x}$.

- For $\sigma \in \mathcal{W} \cup \mathcal{I}(\mathcal{W})$ the path $\mathbb{X}_{z,x}\sigma$ plays the role of $(\Pi_x \sigma)(z)$ or equivalently $(\mathbf{\Pi}\sigma)(z)$.
- For $\tau \in \mathring{\mathcal{N}}$,
  - $\mathbb{X}_{z,x}\tau$ plays the role of $(\Pi_x \tau)(z)$.
  - $\mathbb{X}_{z,x}\mathcal{I}(\tau)$ plays the role of $(\Pi_x \mathcal{I}(\tau))(z)$ and $\gamma_{z,x}(\mathcal{I}(\tau))$. In particular these two quantities are the same and in our context this means that the definition (4.5.4) is actually compatible with the formula (4.5.8) - see (4.5.12).
- For $\tau \in \widetilde{\mathcal{N}}$ and $1 \leq i \leq d$, $\mathbb{X}_{z,x}\mathcal{I}_i^+(\tau)$ plays the role of $\gamma_{z,x}(\mathcal{I}_i(\tau))$ which in general has a different value than $(\Pi_x \mathcal{I}_i(\tau))(z)$. This is why we cannot define $\mathbb{X}_{z,x}\mathcal{I}_i^+(\tau)$ with some formula that is analogous to (4.5.4).

4.5.2 Properties of paths and centrings

The first property we will investigate is Chen's relation.

Definition 4.5.4. *We say a local product satisfies Chen's relation on $\sigma \in \mathcal{T}$ if, for every $z, u, x \in \mathbb{R} \times \mathbb{R}^d$,*

$$(\mathbb{X}_{z,u} \otimes \mathbb{X}_{u,x})\Delta\sigma = \mathbb{X}_{z,x}\sigma \ . \tag{4.5.10}$$

Remark 4.5.5. We use Chen's relation to study the change of base-point operation for tree expansions, and the sole role of $\mathcal{I}_i^+(\tau)$ for $\tau \in \widetilde{\mathcal{N}}$ is to describe this procedure.

Therefore we are not interested in Chen's relation (4.5.10) for the case where $\sigma = \mathcal{I}_i^+(\tau)$ and instead $\mathcal{I}_i^+(\tau)$ plays the role of an intermediate object in the expansion of (4.5.10).

The following lemma is straightforward because of the trivial structure of the coproduct in those cases.

Lemma 4.5.6. *Any local product automatically satisfies Chen's relation on every $\sigma \in \mathcal{W} \cup \mathcal{I}(\mathcal{W}) \cup \mathcal{I}(\mathrm{Poly}) \cup \{\mathcal{I}_i^+(\mathbf{X}_i)\}_{i=1}^d$.*

We also define semi-norms to capture our notion of order bounds, using the convolution with a approximation of unity denoted by $(\cdot)_L$ as introduced in equation (4.1.5).

Definition 4.5.7. *Given a local product $\mathbb{X}$ and $\sigma \in \mathcal{T}_+$, we define*

$$[\mathbb{X};\sigma] := \begin{cases} \sup\limits_{x\in\mathbb{R}\times\mathbb{R}^d} \sup\limits_{L\in(0,1]} \left|\left(\mathbb{X}_{\bullet,x}\sigma\right)_L(x)\right| L^{-|\sigma|} & \text{for } \sigma \in \mathcal{T}_r \cup \mathcal{I}(\mathcal{W}) \ , \\ \sup\limits_{\substack{z,x\in\mathbb{R}\times\mathbb{R}^d \\ d(z,x)\in(0,1)}} |\mathbb{X}_{z,x}\sigma| d(z,x)^{-|\sigma|} & \text{for } \sigma \in \mathcal{T}^{\mathrm{rec}} \ . \end{cases} \tag{4.5.11}$$

We say $\mathbb{X}$ satisfies an order bound on σ if $[\mathbb{X};\sigma] < \infty$.

Remark 4.5.8. We note that for any $\sigma \in \mathcal{I}(\mathrm{Poly}) \cup \{\mathcal{I}_i^+(\mathbf{X}_i)\}_{i=1}^d$ we have the bound $[\mathbb{X};\sigma] \lesssim 1$ uniformly over local products $\mathbb{X}$.

Since we are working in the smooth setting, it is also true that any local product $\mathbb{X}$ satisfies an order bound on $\tau \in \mathcal{T}_r$ (and $\mathcal{I}(\tau) \in \mathcal{I}(\mathcal{W})$). However, it is not obvious and in general not true, that these bounds remain finite, when one passes to the rough limit, where ξ is genuinely only a $C^{-3+\delta}$ distribution. In the application to stochastic PDE, these bounds can be controlled in the limit, but this requires additional probabilistic arguments as well as a renormalization procedure.

Since we have Lemma 4.5.6 and Remark 4.5.8 our goal for this section is to verify that our definitions automatically guarantee that any local product satisfies

- Chen's relation on any $\tau \in \mathcal{I}(\mathring{\mathcal{N}}) \cup \mathring{\mathcal{N}}$, and
- a quantitative order bound on any

$$\tau \in \mathcal{I}(\mathring{\mathcal{N}}) \cup \{\mathcal{I}_i^+(\tau) : 1 \le i \le d,\ \tau \in \widetilde{\mathcal{N}}\} \cup \mathcal{I}(\mathcal{W})$$

  in terms of order bounds on $\tau \in \mathring{\mathcal{N}} \cup \mathcal{T}_r$.

We now turn to showing the desired statements about Chen's relation.

Proving Chen's relation

It is useful to introduce a stronger, partially factorized version of Chen's relation.

Definition 4.5.9. *Given $\mathcal{I}(\tau) \in \mathcal{I}(\mathcal{N})$ we say a local product $\mathbb{X}$ satisfies the strong Chen relation on $\mathcal{I}(\tau)$ if, for every $x, y \in \mathbb{R}^d$, one has the identity*

$$\mathbb{X}_y^{\mathrm{rec}}\mathcal{I}(\tau) = \left(\mathbb{X}_x^{\mathrm{rec}} \otimes \mathbb{X}_{x,y}\right)\Delta\mathcal{I}(\tau). \tag{4.5.12}$$

We remark that it is trivial to check that any local product satisfies the strong Chen relation on $\mathcal{I}(\tau) \in \mathcal{I}(\mathrm{Poly})$. The following lemma is half of our inductive step for proving Chen's relation.

Lemma 4.5.10. *Suppose a local product $\mathbb{X}$ satisfies Chen's relation on $\tau \in \mathring{\mathcal{N}}$, then $\mathbb{X}$*

satisfies the strong Chen relation on $\mathcal{I}(\tau)$.

Proof. Expanding both sides of (4.5.12) gives

$$\begin{aligned}
&\mathcal{L}^{-1}\big(\mathbb{X}_{\bullet,y}\tau\big)(y) + y_j \mathbb{X}_y^{\mathrm{rec}} \mathcal{I}_j^+(\tau) \\
=&\mathcal{L}^{-1}(\mathbb{X}_{\bullet,y}\tau)|_x^y + (y_i - x_i)\mathbb{X}_y^{\mathrm{rec}} \mathcal{I}_i^+(\tau) \\
&+ x_k\big(\mathbb{X}_y^{\mathrm{rec}} \mathcal{I}_k^+(\tau) - \sum_{\widetilde{\tau}\in\mathring{\mathcal{N}}} (\mathbb{X}_x^{\mathrm{rec}} \mathcal{I}_k^+(\widetilde{\tau}))\mathbb{X}_{x,y} C_+(\widetilde{\tau},\tau)\big) \\
&- \sum_{\widetilde{\tau}\in\mathring{\mathcal{N}}} \big(\mathbb{X}_x^{\mathrm{rec}} \mathcal{I}(\widetilde{\tau})\big)\big(\mathbb{X}_{x,y} C_+(\widetilde{\tau},\tau)\big)
\end{aligned} \tag{4.5.13}$$

Each of the three lines on the right-hand side above come from one of the three terms on the right-hand side of the first line of (4.4.3).

Doing the explicit cancellations lets us simplify (4.5.13) to

$$\begin{aligned}
0 =&\mathcal{L}^{-1}(\mathbb{X}_{\bullet,y}\tau)(x) + x_k \sum_{\widetilde{\tau}\in\mathring{\mathcal{N}}} (\mathbb{X}_x^{\mathrm{rec}} \mathcal{I}_k^+(\widetilde{\tau}))\mathbb{X}_{x,y} C_+(\widetilde{\tau},\tau) \\
&+ \sum_{\widetilde{\tau}\in\mathring{\mathcal{N}}} \big(\mathbb{X}_x^{\mathrm{rec}} \mathcal{I}(\widetilde{\tau})\big)\big(\mathbb{X}_{x,y} C_+(\widetilde{\tau},\tau)\big)
\end{aligned} \tag{4.5.14}$$

We then obtain (4.5.14) by using our assumption on Chen's relation for τ to write

$$\begin{aligned}
&\mathcal{L}^{-1}(\mathbb{X}_{\bullet,y}\tau)(x) = \mathcal{L}^{-1}\Big((\mathbb{X}_{\bullet,x}\otimes\mathbb{X}_{x,y})\Delta\tau\Big)(x) \\
&= \sum_{\widetilde{\tau}\in\mathring{\mathcal{N}}} \big(\mathcal{L}^{-1}\mathbb{X}_{\bullet,x}(\widetilde{\tau})\big)(x)\big(\mathbb{X}_{x,y} C_+(\widetilde{\tau},\tau)\big)\ ,
\end{aligned}$$

and then recalling that for any $\widetilde{\tau}$ in the above sum one has

$$\mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\widetilde{\tau})(x) = -\mathbb{X}_x^{\mathrm{rec}} \mathcal{I}(\widetilde{\tau}) - x_k \mathbb{X}_x^{\mathrm{rec}} \mathcal{I}_k^+(\widetilde{\tau})\ .$$

□

The following lemma is the second half of our inductive step.

Lemma 4.5.11. *Fix $\tau \in \mathring{\mathcal{N}}$ and suppose $\mathbb{X}$ is a local product that satisfies the strong Chen property on $\mathcal{I}(\overline{\tau})$ for every $\overline{\tau} \subsetneq \tau$. Then $\mathbb{X}$ satisfies Chen's relation on τ.*

Proof. We have

$$\begin{aligned}(\mathbb{X}_{x,y} \otimes \mathbb{X}_{y,z})\Delta\tau =&(\mathbb{X}_x \otimes \mathbb{X}_y^{\mathrm{rec}} \otimes \mathbb{X}_{y,z})(\Delta \otimes \mathrm{Id})\Delta\tau \\ =&(\mathbb{X}_x \otimes \mathbb{X}_y^{\mathrm{rec}} \otimes \mathbb{X}_{y,z})(\mathrm{Id} \otimes \Delta)\Delta\tau \\ =&(\mathbb{X}_x \otimes \mathbb{X}_z^{\mathrm{rec}})\Delta\tau \\ =&\mathbb{X}_{x,z}\tau\ .\end{aligned}$$

In the first equality we used our identity (4.5.7) for $\mathbb{X}_{x,y}$ and in the second we used the co-associativity property of Lemma 4.4.6.

For the third equality we used the fact that Δ is multiplicative over forests of planted trees so we can use either Lemma 4.5.10 or (4.5.8) for the planted trees that appear in the forests that appear on the right factor of $\Delta\tau$. Fix $\widetilde{\tau} \notin \{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$. Then for any $\mathcal{I}(\overline{\tau}) \in C_+(\widetilde{\tau}, \tau)$ one has $\overline{\tau} \subsetneq \tau$ so one can use Lemma 4.5.10 for these factors. For the factors $\mathcal{I}_i^+(\overline{\tau}) \in C_+(\widetilde{\tau}, \tau)$ one can just use (4.5.8). □

Putting together these two lemmas for our inductive step, combined with Lemma 4.5.6 which gives us the bases cases for our induction, we arrive at the following proposition.

Proposition 4.5.12. *Any local product $\mathbb{X}$ satisfies Chen's relation on $\mathcal{T}$.*

Order bound

Below, for any local product $\mathbb{X}$ and forest of planted trees $\sigma_1 \cdots \sigma_n$, $n \in \mathbb{Z}_{\geq 0}$, we write

$$[\mathbb{X}; \sigma_1 \cdots \sigma_n] := \prod_{j=1}^{n} [\mathbb{X}; \sigma_j]\ .$$

We also write $[\mathbb{X}; 0] := 0$. With this notation we can state the following lemma.

Lemma 4.5.13. *For any $\tau \in \mathcal{T}_r$ and uniform over local products $\mathbb{X}$ one has the estimate*

$$[\mathbb{X}; \mathcal{I}(\tau)] \lesssim \begin{cases} [\mathbb{X}; \tau] & \text{for } \tau \in \mathcal{W}\ , \\ [\mathbb{X}; \tau] + \max\limits_{\overline{\tau} < \tau} [\mathbb{X}; \overline{\tau}][\mathbb{X}; C_+(\overline{\tau}, \tau)] & \text{for } \tau \in \mathring{\mathcal{N}}\ . \end{cases} \tag{4.5.15}$$

Suppose $\tau \in \widetilde{\mathcal{N}}$, then, for any $1 \leq i \leq d$, and uniform over local products, one has

$$[\mathbb{X}; \mathcal{I}_i^+(\tau)] \lesssim [\mathbb{X}; \tau] + \max_{\overline{\tau} < \tau} [\mathbb{X}; \overline{\tau}][\mathbb{X}; C_+(\overline{\tau}, \tau)] \tag{4.5.16}$$

It follows that any $\mathbb{X}$ satisfies an order bound for any $\sigma \in \mathcal{T}_+$.

Proof. We start with proving (4.5.15) for any $\tau \in \mathcal{T}_r$. Clearly the bound is trivial when the corresponding right-hand side of (4.5.15) is infinite so we assume that they are finite.

When $\tau \in \mathcal{W}$ the desired estimate follows from the Schauder Lemma 3.2.1 where we set $U(\bullet, x) = \mathbb{X}_{\bullet,x}\mathcal{I}(\tau) = \mathbb{X}_{\bullet}\mathcal{I}(\tau)$ and where we can take $M \lesssim [\mathbb{X}; \tau]$ by Lemma 3.5.1, which gives an estimate on the multiplication by smooth cutoff function ρ in (4.3.5).

For $\tau \in \mathring{\mathcal{N}}$ we will instead appeal to Lemma 3.2.2. We set

$$U(y, x) = \mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(y) - \mathcal{L}^{-1}(\mathbb{X}_{\bullet,x}\tau)(x) \, . \tag{4.5.17}$$

It is clear that we can take $M^{(1)} \lesssim [\mathbb{X}; \tau]$ for the assumption (3.2.35).

We now verify the three point continuity condition (3.2.38). The role of $\lambda(\bullet, \bullet)$ in (3.2.35) will be played by the quantity $\lambda_\tau(y, x) = (\lambda_\tau^{(i)}(y, x))_{i=1}^d \in \mathbb{R}^d$ where

$$\lambda_\tau^{(i)}(y, x) = -\sum_{\substack{\overline{\tau} \in \mathring{\mathcal{N}} \\ \overline{\tau} \neq \tau}} (\mathbb{X}_y^{\mathrm{rec}} \mathcal{I}_i^+(\overline{\tau})) \mathbb{X}_{y,x} C_+(\overline{\tau}, \tau) \, . \tag{4.5.18}$$

We then write

$$\begin{aligned}
&U(z, x) - U(y, x) - U(z, y) - (z_i - y_i)\lambda_\tau^{(i)}(y, x) \\
=&(\mathcal{L}^{-1}\mathbb{X}_{\bullet,x}\tau)|_y^z - (\mathcal{L}^{-1}\mathbb{X}_{\bullet,y}\tau)|_y^z - (z_i - y_i)\lambda_\tau^{(i)}(y, x) \\
=&(\mathcal{L}^{-1}(\mathbb{X}_{\bullet,y} \otimes \mathbb{X}_{y,x})\Delta\tau)|_y^z - (\mathcal{L}^{-1}\mathbb{X}_{\bullet,y}\tau)|_y^z - (z_i - y_i)\lambda_\tau^{(i)}(y, x) \\
=&\sum_{\substack{\overline{\tau} \in \mathring{\mathcal{N}} \\ \overline{\tau} \neq \tau}} (\mathcal{L}^{-1}(\mathbb{X}_{\bullet,y}\overline{\tau}))|_y^z \mathbb{X}_{y,x} C_+(\overline{\tau}, \tau) - (z_i - y_i)\lambda_\tau^{(i)}(y, x) \\
=&\sum_{\substack{\overline{\tau} \in \mathring{\mathcal{N}} \\ \overline{\tau} \neq \tau}} \mathbb{X}_{z,y}\mathcal{I}(\overline{\tau})\mathbb{X}_{y,x} C_+(\overline{\tau}, \tau).
\end{aligned} \tag{4.5.19}$$

For the second equality above we appealed to Proposition 4.5.12 to use Chen's relation for τ. Then by inserting the order bound for every term in the last line of (4.5.19) we see that in (3.2.35) we can take $M^{(2)} \lesssim \max_{\overline{\tau} < \tau} [\mathbb{X}; \overline{\tau}][\mathbb{X}; C_+(\overline{\tau}, \tau)]$.

We turn to proving (4.5.16) and so we fix $\tau \in \widetilde{\mathcal{N}}$. We obtain the desired estimate by applying Lemma 3.2.4. Here we again define $U(\bullet, \bullet)$ as in (4.5.17) and $\lambda_\tau(\bullet, \bullet)$ by (4.5.18). Thanks to the computation (4.5.19) we see we can take $M = \max_{\overline{\tau} < \tau} [\mathbb{X}; \overline{\tau}][\mathbb{X}; C_+(\overline{\tau}, \tau)]$ in (3.2.41). We also note that ν_τ is then the optimal ν referenced in Lemma 3.2.4 and that we have

$$\mathbb{X}_{y,x}\mathcal{I}_i^+(\tau) = \nu_\tau^{(i)}(y) - \nu_\tau^{(i)}(x) + \lambda_\tau^{(i)}(y, x)$$

and so the desired estimate is given by (3.2.42). □

4.6 Modelled distribution

With the definition of local products and their associated paths in place, we now show how to use them to give a good local approximation to the solution v of the remainder equation. As explained above in (4.2.18), we seek a local approximation to v of the form

$$v(y) \approx \mathbb{X}_{y,x}\Theta(x) = \sum_{\tau\in\mathcal{N}} \Theta_x(\tau)\mathbb{X}_{y,x}\mathcal{I}(\tau), \tag{4.6.1}$$

for suitable coefficients $\Theta \in C^\infty(\mathbb{R}\times\mathbb{R}^d; \mathrm{Vec}(\mathcal{N}))$ (which we interchangeably view as a map $\Theta : \mathcal{N} \to C^\infty$ taking $\tau \mapsto \Theta_\bullet(\tau)$) and with an error of order $\lesssim d(x,y)^\gamma$. In this section we first introduce a family of seminorms that measure the regularity of the coefficient map Θ and that ultimately permits us to bound renormalized products. Subsequently, we turn to a specific choice of coefficients Θ (denoted by Υ, see Definition 4.6.6) which arise in "freezing of coefficient procedure" described in Section 4.2.4. The main result of this section, Theorem 4.6.10, shows a close connection between the various seminorms for this specific choice of coefficient.

In order to motivate the regularity condition we rewrite equation (4.6.1) for another base-point $\overline{x}$ (but for the same argument y)

$$v(y) \approx \sum_{\overline{\tau}\in\mathcal{N}} \Theta_{\overline{x}}(\overline{\tau})\mathbb{X}_{y,\overline{x}}\mathcal{I}(\overline{\tau}), \tag{4.6.2}$$

then use Chen's relation (4.5.10) and Lemma 4.4.13 in the form

$$\mathbb{X}_{y,\overline{x}}\mathcal{I}(\overline{\tau}) = \sum_{\tau\in\mathcal{N}} \mathbb{X}_{y,x}\mathcal{I}(\tau)\mathbb{X}_{x,\overline{x}}C_+(\tau,\overline{\tau})$$

to rewrite the right-hand side of (4.6.2) and compare the resulting expression to (4.6.1), arriving at

$$\Big|\sum_{\tau\in\mathcal{N}}\Big(\Theta_x(\tau) - \sum_{\overline{\tau}\in\mathcal{N}}\Theta_{\overline{x}}(\overline{\tau})\mathbb{X}_{x,\overline{x}}C_+(\tau,\overline{\tau})\Big)\mathbb{X}_{y,x}\mathcal{I}(\tau)\Big| \lesssim d(x,y)^\gamma + d(\overline{x},y)^\gamma. \tag{4.6.3}$$

Specialising this inequality to those y for which $d(\overline{x},y) \approx d(x,y) \approx d(x,\overline{x}) \approx d$ yields the estimate

$$\Big|\sum_{\tau\in\mathcal{N}}\Big(\Theta_x(\tau) - \sum_{\overline{\tau}\in\mathcal{N}}\Theta_{\overline{x}}(\overline{\tau})\mathbb{X}_{x,\overline{x}}C_+(\tau,\overline{\tau})\Big)\mathbb{X}_{y,x}\mathcal{I}(\tau)\Big| \lesssim d^\gamma. \tag{4.6.4}$$

In view of the the order bound (4.5.11)

$$|\mathbb{X}_{y,x}\mathcal{I}(\tau)| \lesssim d^{|\tau|+2},$$

the following definition is natural.

Definition 4.6.1. *Let $\mathbb{X}_\bullet$ be a local product and $\mathbb{X}_{\bullet,\bullet}$ be the path constructed from $\mathbb{X}_\bullet$. Then for $\Theta \in C^\infty(\mathbb{R} \times \mathbb{R}^d; \mathrm{Vec}(\mathcal{N}))$ for $\tau \in \mathcal{N}$ and $0 < \gamma < 2$ we define*

$$U^\tau_{\gamma-2}(y,x) := \Theta_y(\tau) - \sum_{\substack{\overline{\tau}\in\mathcal{N} \\ |\overline{\tau}|<\gamma-2}} \Theta_x(\overline{\tau})\mathbb{X}_{y,x}C_+(\tau,\overline{\tau}), \tag{4.6.5}$$

and the seminorm

$$[U^\tau]_{\gamma-|\tau|-2} := \sup_{x,y} \frac{1}{d(x,y)^{\gamma-|\tau|-2}} \big|U^\tau_{\gamma-2}(y,x)\big|. \tag{4.6.6}$$

It is important to observe that the semi-norm $[U^\tau]_{\gamma-|\tau|-2}$ involves the coefficients, $\Theta_{\overline{\tau}}$ as well as the paths $\mathbb{X}_{\bullet,\bullet}$ on all symbols $\overline{\tau}$ for which $C_+(\tau,\overline{\tau})$ does not vanish, and that all of these trees $\overline{\tau}$ satisfy $\tau \leqslant \overline{\tau}$. Also, for $\tau = \mathbf{1}$, in view of the identity $C_+(\mathbf{1},\overline{\tau}) = \overline{\tau}$ and $|\mathbf{1}| = -2$ the quantity $[U^{\mathbf{1}}]_\gamma$ measures exactly the size of the error in the expression (4.6.1) at the beginning of this discussion.

Remark 4.6.2. The definition of the semi-norm corresponds exactly to Hairer's definition of a *modelled distribution*, [41, Definition 3.1]. In Hairer's notation the expression $|U^\tau_{\gamma-2}(y,x)|$ becomes

$$\|\Theta(x) - \Gamma_{xy}\Theta(y)\|_{|\mathcal{I}(\tau)|} .$$

The following lemma relates the notion of classical derivative with the generalised derivatives that appear in the modeled distribution.

Lemma 4.6.3. *Let $1 < \gamma < 2$. Fix a local product $\mathbb{X}$ and $\Theta \in C^\infty(\mathbb{R} \times \mathbb{R}^d; \mathrm{Vec}(\mathcal{N}))$ with the property that, with $U^{\mathbf{1}}_{\gamma-2}(y,x)$ defined as in* (4.6.5)*, we have $[U^{\mathbf{1}}]_\gamma < \infty$. Then, for $1 \le i \le d$,*

$$\Theta_x(\mathbf{X}_i) = \partial_i\Big(\Theta_y(\mathbf{1}) - \sum_{\substack{\overline{\tau}\in\mathcal{N} \\ |\overline{\tau}|<-1}} \Theta_x(\overline{\tau})\mathbb{X}_{y,x}C_+(\mathbf{1},\overline{\tau})\Big)\Big|_{y=x} , \tag{4.6.7}$$

where the partial derivative ∂_i acts in the variable y.

Proof. Note that by assumption we have that that $|U^{\mathbf{1}}_{\gamma-2}(y,x)| \lesssim d(y,x)^\gamma$ and since $\gamma > 1$ it follows that

$$\big(\partial_i U^{\mathbf{1}}_{\gamma-2}(y,x)\big)\big|_{y=x} = 0 .$$

We obtain the desired result by plugging in the definition of $U^{\mathbf{1}}_{\gamma-2}(y,x)$ and recalling

that

$$\begin{aligned}\big(\partial_i \mathbb{X}_{y,x} C_+(\mathbf{1},\mathbf{1})\big)\big|_{y=x} &= \big(\partial_i \mathbb{X}_{y,x}\mathcal{I}(\mathbf{1})\big)\big|_{y=x} = 0\ ,\\ \big(\partial_i \mathbb{X}_{y,x} C_+(\mathbf{1},\mathbf{X}_j)\big)\big|_{y=x} &= \big(\partial_i \mathbb{X}_{y,x}\mathcal{I}(\mathbf{X}_j)\big)\big|_{y=x} = \delta_{\{j=i\}}\ ,\\ \text{and } \big(\partial_i \mathbb{X}_{y,x} C_+(\mathbf{1},\overline{\tau})\big)\big|_{y=x} &= 0 \text{ for } \overline{\tau} \in \mathring{\mathcal{N}} \text{ with } |\overline{\tau}| > -1\ .\end{aligned}$$

In the last statement we are using that $|\mathbb{X}_{y,x} C_+(\mathbf{1},\overline{\tau})| \lesssim d(y,x)^{|\overline{\tau}|+2}$. □

We now introduce some short-hand notation that will be very useful in the following calculations. First, for a given local product $\mathbb{X}$, for $\Theta \in C^\infty(\mathbb{R}\times\mathbb{R}^d;\mathrm{Vec}(\mathcal{N}))$ and $\gamma \in (0,2)$ we denote by

$$V_\gamma(y,x) := \sum_{\substack{\tau\in\mathcal{N},\\ |\tau|<\gamma-2}} \Theta_x(\tau)\mathbb{X}_{y,x}\mathcal{I}(\tau)\ . \tag{4.6.8}$$

We also introduce a truncated "square" and a "spatial derivative":

$$V_\gamma^2(y,x) := \sum_{\substack{\tau_1,\tau_2\in\mathcal{N},\\ |\tau_1|+|\tau_2|<\gamma-4}} \Theta_x(\tau_1)\Theta_x(\tau_2)\mathbb{X}_{y,x}\mathcal{I}(\tau_1)\mathbb{X}_{y,x}\mathcal{I}(\tau_2)\ , \tag{4.6.9}$$

$$V_\gamma^{(i)}(y,x) := \sum_{\substack{\tau\in\widetilde{\mathcal{N}},\\ |\tau|<\gamma-1}} \Theta_x(\tau)\mathbb{X}_{y,x}\mathcal{I}_i^+(\tau)\ . \tag{4.6.10}$$

Note that due to the choice of index set $V_\gamma^2(y,x)$ does not coincide with the point-wise square $(V_\gamma(y,x))^2$. Note furthermore, that recalling the definition (4.6.5) specialised to $\tau = \mathbf{X}_i$ as well as the identity $C_+(\mathbf{X}_i,\tau) = \mathcal{I}_i^+(\tau)$ for $\tau \in \widetilde{\mathcal{N}}$ and $= 0$ otherwise (see Table 4.1) we have the identity

$$U_\gamma^{\mathbf{X}_i}(y,x) = \Theta_y(\mathbf{X}_i) - V_{\gamma-1}^{(i)}(y,x). \tag{4.6.11}$$

A first nice observation is a control for the "three point continuity operator" for V_γ (the left-hand side of (4.6.12) below) in terms of the $U_{\gamma-2}^\tau$ and $\mathbb{X}_{\bullet,\bullet}$. This "three point continuity operator" corresponds exactly to Gubinelli's δ operator [33, 34]. In our calculations this quantity is needed to bound derivatives (see (4.6.16) below) and as input to the Schauder lemmas presented in Section 3.2.4.

Lemma 4.6.4. *Let $\mathbb{X}_\bullet$ be a local product. Let $\Theta \in C^\infty(\mathbb{R}\times\mathbb{R}^d;\mathrm{Vec}(\mathcal{N}))$ and let V be defined as in* (4.6.8)*. Then for any space-time points $x,y,z \in \mathbb{R}\times\mathbb{R}^d$ we have*

$$V_\gamma(z,x) - V_\gamma(z,y) + V_\gamma(y,y) - V_\gamma(y,x)$$

$$= - \sum_{\substack{\tau \in \mathcal{N} \setminus \{\mathbf{1}\}, \\ |\tau| < \gamma - 2}} U^{\tau}_{\gamma-2}(y,x)\, \mathbb{X}_{z,y}\mathcal{I}(\tau)\ . \tag{4.6.12}$$

Proof. We reorganise the terms on the left-hand side of (4.6.12) to write

$$\begin{aligned} V_\gamma(z,x) - V_\gamma(z,y) + V_\gamma(y,y) - V_\gamma(y,x) \\ = \sum_{\substack{\overline{\tau} \in \mathcal{N}, \\ |\overline{\tau}| < \gamma - 2}} \Theta_x(\overline{\tau}) \big(\mathbb{X}_{z,x}\mathcal{I}(\overline{\tau}) - \mathbb{X}_{y,x}\mathcal{I}(\overline{\tau}) \big) \\ - \sum_{\substack{\tau \in \mathcal{N}, \\ |\tau| < \gamma - 2}} \Theta_y(\tau) \big(\mathbb{X}_{z,y}\mathcal{I}(\tau) - \mathbb{X}_{y,y}\mathcal{I}(\tau) \big). \end{aligned} \tag{4.6.13}$$

We use Chen's relation (4.5.10) for the terms in the first sum on the right-hand

$$\mathbb{X}_{z,x}\mathcal{I}(\overline{\tau}) - \mathbb{X}_{y,x}\mathcal{I}(\overline{\tau}) = \sum_{\substack{\tau \in \mathcal{N} \\ |\tau| < \gamma - 2}} \big(\mathbb{X}_{z,y}\mathcal{I}(\tau) - \mathbb{X}_{y,y}\mathcal{I}(\tau) \big) \mathbb{X}_{y,x} C_+(\tau, \overline{\tau})\ .$$

Plugging this into the first term on the right-hand side of (4.6.13), exchanging the summation in τ and $\overline{\tau}$ gives

$$\begin{aligned} V_\gamma(z,x) - V_\gamma(z,y) + V_\gamma(y,y) - V_\gamma(y,x) = \\ \sum_{\substack{\tau \in \mathcal{N}, \\ |\tau| < \gamma - 2}} \Big(\big(\mathbb{X}_{z,y}\mathcal{I}(\tau) - \mathbb{X}_{y,y}\mathcal{I}(\tau) \big) \big(\sum_{\substack{\overline{\tau} \in \mathcal{N}, \\ |\overline{\tau}| < \gamma - 2}} \Theta_x(\overline{\tau}) \mathbb{X}_{y,x} C_+(\tau, \overline{\tau}) - \Theta_y(\tau) \big) \Big). \end{aligned}$$

Finally, noting that $\mathbb{X}_{z,y}\mathcal{I}(\mathbf{1}) - \mathbb{X}_{y,y}\mathcal{I}(\mathbf{1}) = 0$ and $\mathbb{X}_{y,y}\mathcal{I}(\tau) = 0$ for $\tau \in \mathcal{N} \setminus \{\mathbf{1}\}$ leads to the desired expression (4.6.12). □

The following lemma gives relations between V, V^2 and $V^{(i)}$. They will be used heavily in Section 4.10.

Lemma 4.6.5. *Truncation: for $0 < \beta < \gamma < 2$, $V_{\gamma-\beta}$ is a truncation of V_γ*

$$\begin{aligned} \Theta_y(\mathbf{1}) - V_\gamma(y,x) = & \Theta_y(\mathbf{1}) - V_{\gamma-\beta}(y,x) \\ & - \sum_{\gamma-\beta-2 \leqslant |\tau| < \gamma-2} \Theta_x(\tau) \mathbb{X}_{y,x}\mathcal{I}(\tau)\ . \end{aligned} \tag{4.6.14}$$

Multiplication: for $0 < \gamma < 1$, V^2_γ is a truncation of $(V_\gamma)^2$:

$$\begin{aligned} \Theta_y(\mathbf{1})^2 - V^2_\gamma(y,x) = \Theta_y(\mathbf{1})(\Theta_y(\mathbf{1}) - V_\gamma(y,x)) \\ + \sum_{-2 \leqslant |\tau| < \gamma - 2} \Theta_x(\tau) \mathbb{X}_{y,x}\mathcal{I}(\tau)(\Theta_y(\mathbf{1}) - V_{\gamma-|\tau|-2}(y,x)). \end{aligned} \tag{4.6.15}$$

Derivative: To control derivatives we use the following reorganisation of Lemma 4.6.4: for $1 < \gamma < 2$ *we have*

$$V_\gamma(z,x) - V_\gamma(z,y) + V_\gamma(y,y) - V_\gamma(y,x) \tag{4.6.16}$$
$$+ \sum_{i=1}^{d} \big(\Theta_y(\mathbf{X}_i) - V^{(i)}_{\gamma-1}(y,x)\big)(y-z)_i = - \sum_{\substack{\tau \in \mathring{\mathcal{N}}, \\ |\tau| < \gamma - 2}} U^\tau_{\gamma-2}(y,x)\, \mathbb{X}_{z,y}\mathcal{I}(\tau).$$

This last identity (4.6.16) will be combined with Lemma 3.2.5 to give a bound on $V^{(i)}$ below.

Proof. The first two identities are immediate. In the third one, we use the identity (4.6.11) and rewrite the term corresponding to $\tau = \mathbf{X}_i$, for which $\mathbb{X}_{z,y}\mathbf{X}_i = (y-z)_i$. □

We introduce in the following definition a coefficient map Υ depending on some real valued functions $v_{\mathbf{1}}$ and $v_{\mathbf{X}_i}$, $i = 1...d$, on $\mathbb{R} \times \mathbb{R}^d$.

Definition 4.6.6. *Given real parameters* $v_{\mathbf{1}},\ v_{\mathbf{X}_i},\ i = 1...d$ *and* $\tau \in \mathcal{N} \sqcup \mathcal{W}$ *we set*

$$\Upsilon(\tau)[v_{\mathbf{1}}, v_{\mathbf{X}}] := \begin{cases} v_{\mathbf{1}}, & \tau = \mathbf{1}, \\ v_{\mathbf{X}_i}, & \tau = \mathbf{X}_i, \\ 1, & \tau = \Xi, \\ -\prod_{i=1}^3 \Upsilon(\tau_i)[v_{\mathbf{1}}, v_{\mathbf{X}}], & \tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3), \end{cases} \tag{4.6.17}$$

where we adopt, above and in what follows, the notational convention $v_{\mathbf{X}} = (v_{\mathbf{X}_i})_{i=1}^d$. *We may omit the parameters* $[v_{\mathbf{1}}, v_{\mathbf{X}}]$ *from the notation when there is no possible confusion, We usually work in the case where* $v_{\mathbf{1}}$ *and* $v_{\mathbf{X}}$ *are functions of spacetime* $\mathbb{R} \times \mathbb{R}^d$ *in which case we use the shorthand:* $\Upsilon(\tau)[v_{\mathbf{1}}(z), v_{\mathbf{X}}(z)] =: \Upsilon_z(\tau)$.

We extend Υ *to planted trees, in particular we set* $\Upsilon(\mathcal{I}(\tau)) := \Upsilon(\tau)$ *for* $\tau \in \mathcal{W} \cup \mathcal{N}$ *and* $\Upsilon(\mathcal{I}_i^+(\tau)) = \Upsilon(\tau)$ *for* $\tau \in \widetilde{\mathcal{N}} \cup \{\mathbf{X}_i\}_{i=1}^d$. *We also extend* Υ *to forests of planted trees by multiplicativity.*

It is straightforward to derive an explicit formula from the earlier recursive formula Υ, we state this as a lemma.

Lemma 4.6.7.

$$\Upsilon(\tau)[v_{\mathbf{1}}, v_{\mathbf{X}}] = (-1)^{\frac{m(\tau)-1}{2}} v_{\mathbf{1}}^{m_{\mathbf{1}}(\tau)} \prod_{i=1}^d v_{\mathbf{X}_i}^{m_{\mathbf{x}_i}(\tau)}.$$

Note that since we only consider trees of negative order, we always have $m_{\mathbf{1}}(\tau) + 2m_{\mathbf{x}}(\tau) < 3$. In particular only $\pm v_{\mathbf{1}}$, $\pm v_{\mathbf{1}}^2$, $\pm v_{\mathbf{X}_i}$ or ± 1 can appear and these possibilities

correspond, respectively, to $m_{\mathbf{1}}(\tau) = 1$, $m_{\mathbf{1}}(\tau) = 2$, $m_{\mathbf{x}_i}(\tau) = 1$, and $m_{\mathbf{1}}(\tau) = m_{\mathbf{x}}(\tau) = 0$.

Assumption 4.6.8. *For the remainder of the thesis, we will always assume that the coefficient map $\Theta \in C^\infty(\mathbb{R} \times \mathbb{R}^d; \mathrm{Vec}(\mathcal{N}))$ is of the form*

$$\Theta_z(\bullet) = \Upsilon(\bullet)[v_{\mathbf{1}}(z), v_{\mathbf{X}}(z)] = \Upsilon_z(\bullet) \text{ for some } v_{\mathbf{1}},\ v_{\mathbf{X}}\ . \tag{4.6.18}$$

We enforce the relation (4.6.18) *for the rest of the thesis, in particular this is implicit in any use of the notation U^τ.*

The following identities are the main motivation behind our definition of Υ.

Lemma 4.6.9. *One has*

$$-\Big(\sum_{\tau \in \mathcal{N} \cup \mathcal{W}} \Upsilon(\tau)\mathcal{I}(\tau)\Big)^3 = \sum_{\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} \Upsilon(\tau)\tau\ . \tag{4.6.19}$$

We also have for any $\tau \in \mathcal{N}$, $\overline{\tau} \in \mathring{\mathcal{N}}$ such that $C_+(\tau, \overline{\tau}) \neq 0$,

$$\Upsilon(\overline{\tau}) = (-1)^{\frac{m(\tau)-1}{2}} \Upsilon(C_+(\tau, \overline{\tau})), \tag{4.6.20}$$

where Υ acts on forests multiplicatively.

Proof. For the equality (4.6.19) we first note that we have this equality if we dropped all the Υ's and dropped the minus sign on the left-hand side, this is just (4.3.4). What is left is to make sure that by inserting the minus sign and Υ's in (4.6.19), the unplanted trees on either side of the equation have the same coefficient, but this is an immediate consequence of the last line of (4.6.17).

The identity (4.6.20) follows from Lemma 4.6.7 and Lemma 4.4.12. □

The key result of this section is the observation that under the structure assumption described in Remark 4.6.8, all continuity conditions are controlled by the condition on $\mathbf{1}$ and $\mathbf{X}_i$. This follows from the bounds established in Lemma 4.6.5 and the following theorem which uses the structure of Υ.

Theorem 4.6.10. *For $\tau \in \mathcal{T}$, the quantity $U^\tau_\gamma(y, x)$ takes the following form*

$$U^\tau_\gamma(y,x) = \begin{cases} (-1)^{\frac{m(\tau)-1}{2}}(v_{\mathbf{1}}(y) - V_{\gamma-|\tau|}(y,x)) & \text{if } m_{\mathbf{1}}(\tau) = 1, \\ (-1)^{\frac{m(\tau)-1}{2}}(v_{\mathbf{1}}(y)^2 - V^2_{\gamma-|\tau|}(y,x)) & \text{if } m_{\mathbf{1}}(\tau) = 2, \\ (-1)^{\frac{m(\tau)-1}{2}}(v_{\mathbf{X}_i}(y) - V^{(i)}_{\gamma-|\tau|}(y,x)) & \text{if } m_{\mathbf{x}_i}(\tau) = 1, \\ 0 & \text{if } m_{\mathbf{1}}(\tau), m_{\mathbf{x}}(\tau) = 0. \end{cases} \tag{4.6.21}$$

Proof. From Lemma 4.6.9, we have

$$\Upsilon(\overline{\tau})C_+(\tau,\overline{\tau}) = (-1)^{\frac{m(\tau)-1}{2}}\Upsilon(C_+(\tau,\overline{\tau}))C_+(\tau,\overline{\tau}).$$

This allows us to write:

$$U^\tau_\gamma(y,x) = \Upsilon_y(\tau) - (-1)^{\frac{m(\tau)-1}{2}}\sum_{|\overline{\tau}|<\gamma}\Upsilon_x(C_+(\tau,\overline{\tau}))\mathbb{X}_{y,x}C_+(\tau,\overline{\tau}).$$

We can use Lemma 4.4.12 to study the different cases:

- If $m_{\mathbf{1}}(\tau) = 1$ and $C_+(\tau,\overline{\tau}) \neq 0$ then there exists a unique $\widetilde{\tau} \in \mathcal{N}$ such that $C_+(\tau,\overline{\tau}) = \mathcal{I}(\widetilde{\tau})$. Conversely, for each $\widetilde{\tau} \in \mathcal{N}$ with $|\widetilde{\tau}| < \gamma - |\tau| - 2$, there exists a unique $\overline{\tau} \in \mathcal{N}$ with $|\overline{\tau}| < \gamma - 2$ such that $C_+(\tau,\overline{\tau}) = \mathcal{I}(\widetilde{\tau})$. Indexing the sum over this $\widetilde{\tau}$ gives the expression of $V_{\gamma-|\tau|}$.
- If $m_{\mathbf{x}_i}(\tau) = 1$ and $C_+(\tau,\overline{\tau}) \neq 0$ then there exists a unique $\widetilde{\tau} \in \mathcal{N}$ such that $C_+(\tau,\overline{\tau}) = \mathcal{I}_i^+(\widetilde{\tau})$. Indexing the sum over this $\widetilde{\tau}$ gives the expression of $V^{(i)}_{\gamma-|\tau|}$.
- If $m_{\mathbf{1}}(\tau) = 2$ and $C_+(\tau,\overline{\tau}) \neq 0$ then there exists a unique non-commutative couple $(\widetilde{\tau}_1,\widetilde{\tau}_2) \in \mathcal{N}^2$ such that $C_+(\tau,\overline{\tau}) = \mathcal{I}(\widetilde{\tau}_1)\cdot\mathcal{I}(\widetilde{\tau}_2)$. Indexing the sum over these $\widetilde{\tau}_1,\widetilde{\tau}_2$ gives the expression of $V^2_{\gamma-|\tau|}$, using also the multiplicative action of $\mathbb{X}$ on forests of planted trees.

We finally see that we get the correct order using the fact that $|\tau| + |C(\tau,\overline{\tau})| = |\overline{\tau}|$. □

Lemma 4.6.3 above showed that the continuity condition on a modelled distribution enforces the relation (4.6.7) between the coefficients $\Theta_z(\mathbf{X}_i)$ and the other coefficients. Since we are now imposing the structural condition (4.6.18), we see that all the left-hand side of (4.6.7) is given by $v_{\mathbf{X}_i}(x)$ and the right-hand side of (4.6.7) has no dependence on $v_{\mathbf{X}}(x)$ - therefore the continuity condition combined with (4.6.18) determines $v_{\mathbf{X}}(x) = (v_{\mathbf{X}_i}(x))_{i=1}^d$ as a function of $v_{\mathbf{1}}$ and the local product $\mathbb{X}$ along with associated derivatives. For future use we encode this as a map $(v_{\mathbf{1}},\mathbb{X}) \mapsto \mathcal{D}_i^{\mathbb{X}}v_{\mathbf{1}} = v_{\mathbf{X}_i}$.

Definition 4.6.11. *Given a local product $\mathbb{X}$ and a smooth function $v : \mathbb{R}\times\mathbb{R}^d_x \to \mathbb{R}$ we define $\mathcal{D}^{\mathbb{X}}v = (\mathcal{D}_i^{\mathbb{X}}v)_{i=1}^d$, $\mathcal{D}_i^{\mathbb{X}}v : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ by setting, for $1 \leq i \leq d$,*

$$(\mathcal{D}_i^{\mathbb{X}}v)(x) = \sum_{\substack{\overline{\tau}\in\hat{\mathcal{N}}\\ |\overline{\tau}|<-1}}\Upsilon_x(\overline{\tau})\big(\partial_i\mathbb{X}_{y,x}C_+(\mathbf{1},\overline{\tau})\big)\big|_{y=x} - \partial_i v(y)\big|_{y=x},$$

where the partial derivatives above acts in the dummy variable y and the $\Upsilon_x(\cdot)$ coefficients above are defined using the parameter $v(x) = v_{\mathbf{1}}(x)$. Note that we do not need to specify a parameter $v_{\mathbf{X}}(x)$ for the Υ_x map above since every $\overline{\tau}$ appearing in this sum satisfies $m_{\mathbf{x}}(\overline{\tau}) = 0$.

4.7 Renormalised products of tree expansions

In this section we define renormalised "point-wise" products, taking as input a local product and tree expansions. Fix some smooth noise ξ and let $\mathbb{X}$ be a lift of ξ. This means that we enforce that $\mathbb{X}\Xi = \xi$. Upon fixing the choice of $\mathbb{X}$ we will arrive at analog of the (4.1.1) equation which we now try to identify. The solution to this yet to be identified equation will be written in the form

$$\begin{aligned}\phi(z) &= v(z) + \sum_{\tau\in\mathcal{W}} (-1)^{\frac{m(\tau)-1}{2}} \mathbb{X}_z\mathcal{I}(\tau) \\ &= \sum_{\tau\in\mathcal{W}\cup\{\mathbf{1}\}} \Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau) \\ &= \sum_{\tau\in\mathcal{W}\cup\mathcal{N}} \Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau)\ . \end{aligned} \tag{4.7.1}$$

Above, in passing from the first to second line, we have used the definition of Υ on $\mathcal{W}$ in Lemma 4.6.7 as well as the fact that $\mathbb{X}_{z,z}\mathcal{I}(\tau) = \mathbb{X}_z\mathcal{I}(\tau)$ for $\tau \in \mathcal{W}$. In passing to the last line we used the simple observation that for $\tau \in \mathcal{N}\setminus\{\mathbf{1}\}$, $\mathbb{X}_{z,z}\mathcal{I}(\tau) = 0$. This trivially gives the identity

$$\begin{aligned}\phi^3(z) &= \sum_{\tau_1,\tau_2,\tau_3\in\mathcal{W}\cup\mathcal{N}} \Upsilon_z(\tau_1)\Upsilon_z(\tau_2)\Upsilon_z(\tau_3)\mathbb{X}_{z,z}\mathcal{I}(\tau_1)\mathbb{X}_{z,z}\mathcal{I}(\tau_2)\mathbb{X}_{z,z}\mathcal{I}(\tau_3) \\ &= -\sum_{\tau_1,\tau_2,\tau_3\in\mathcal{W}\cup\mathcal{N}} \Upsilon_z(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3))\mathbb{X}_{z,z}\mathcal{I}(\tau_1)\mathbb{X}_{z,z}\mathcal{I}(\tau_2)\mathbb{X}_{z,z}\mathcal{I}(\tau_3)\ . \end{aligned} \tag{4.7.2}$$

The renormalisation now consists of replacing each of the point-wise products $\mathbb{X}_{z,z}\mathcal{I}(\tau_1)$ $\mathbb{X}_{z,z}\mathcal{I}(\tau_2)$ $\mathbb{X}_{z,z}\mathcal{I}(\tau_3)$ which in general we do not control by the terms $\mathbb{X}_{z,z}\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$ which we control by assumption. The following definition extends this idea to more general expansions.

Definition 4.7.1. *Fix a local product $\mathbb{X}$. Suppose we are given, for $1 \le i \le 3$, smooth functions $\theta^{(i)}(z) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $\Theta^{(i)} : \mathbb{R}\times\mathbb{R}^d \to \mathrm{Vec}(\mathcal{T}_l)$ with*

$$\theta^{(i)}(z) = \mathbb{X}_{z,z}\Theta^{(i)}(z) = \sum_{\tau\in\mathcal{N}\cup\mathcal{W}} \Theta_z^{(i)}(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau)\ . \tag{4.7.3}$$

Then, we define

$$\begin{aligned}&(\theta^{(1)} \circ_{\mathbb{X}} \theta^{(2)} \circ_{\mathbb{X}} \theta^{(3)})(z) = \\ &\sum_{\tau=\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)\in\mathring{\mathcal{N}}\cup\mathring{\mathcal{W}}} \Theta_z^{(1)}(\tau_1)\Theta_z^{(2)}(\tau_2)\Theta_z^{(3)}(\tau_3)\mathbb{X}_{z,z}\tau\ . \end{aligned} \tag{4.7.4}$$

In the case where there is a single function $\theta^{(i)} = \theta$ and a single corresponding tree expansion $\Theta^{(i)} = \Theta$ then we just write $\theta^{\circ_{\mathbb{X}}3}$ for the left-hand side of (4.7.4).

We adopt the convention that when $\theta^{(i)}(z) = \mathbb{X}_{z,z}\mathcal{I}(\tau) = \mathbb{X}_z\mathcal{I}(\tau)$, where $\tau \in \mathcal{W}$, appears as a factor in a local product, we will implicitly take $\Theta^{(i)}(z) = \mathcal{I}(\tau)$. We remark that, with the generality we allow in the definition of local products, there is no reason to expect that $(\theta^{(1)} \circ_{\mathbb{X}} \theta^{(2)} \circ_{\mathbb{X}} \theta^{(3)})(z)$ is given by some polynomial in the functions $\theta^{(i)}(z)$ and their spatial derivatives. However, a trivial case where there is such a correspondence is in the case of a multiplicative local product in which case one clearly has $(\theta^{(1)} \circ_{\mathbb{X}} \theta^{(2)} \circ_{\mathbb{X}} \theta^{(3)})(z) = \theta^{(1)}(z)\theta^{(2)}(z)\theta^{(3)}(z)$.

Remark 4.7.2. An important observation about the importance of these tree expansions is the following. In (4.7.3), the contribution on the two right-hand sides from $\tau \in \mathcal{N} \setminus \{\mathbf{1}\}$ vanishes due to the order bound for any local product.

Similarly, if the local product $\mathbb{X}$ is multiplicative then none of the terms involving either τ_1 or τ_2 or $\tau_3 \in \mathcal{N}$ contribute to the value of the renormalised product.

However, if the $\mathbb{X}$ is not multiplicative then it can certainly be the case that these terms from $\tau \in \mathcal{N} \setminus \{\mathbf{1}\}$ contribute to the value of the renormalised product even though they do not contribute to the value of the $\theta^{(i)}(z)$. At the same time, in the end we only need to keep products of trees $\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$ with $|\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)| < 0$ in our analysis of renormalised products. This motivates our truncation convention for tree products described at the end of Section 4.3.1.

We can now specify the equations we obtain a priori bounds for.

Definition 4.7.3. *Fix a local product $\mathbb{X}$. Then we say the solution of the Φ^4 equation driven by $\mathbb{X}$ is a smooth function $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ solving*

$$(\partial_t - \Delta)\phi = \phi^{\circ_{\mathbb{X}}3} + \xi \, . \tag{4.7.5}$$

where we write $\xi = \mathbb{X}_z\Xi$ and the tree expansion Φ for ϕ used to define $\phi^{\circ_{\mathbb{X}}3}$ is defined by

$$\Phi(z) = \sum_{\tau \in \mathcal{N} \cup \mathcal{W}} \Upsilon_z(\tau)\mathcal{I}(\tau) \, ,$$

where Υ is defined as in Definition 4.6.6 and the parameter $v_{\mathbf{1}}(z)$ is given by

$$v_{\mathbf{1}}(z) = \phi(z) - \sum_{\tau \in \mathcal{W}} \Upsilon_z(\tau)\mathbb{X}_z\mathcal{I}(\tau) \, , \tag{4.7.6}$$

while we set parameter $v_{\mathbf{X}} = \mathcal{D}^{\mathbb{X}}v_{\mathbf{1}}$ as given in Definition 4.6.11.

Definition 4.7.4. *Fix a local product $\mathbb{X}$. Then we say the solution of the Φ^4-remainder*

equation driven by $\mathbb{X}$ is a smooth function $v : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies the equation

$$
\begin{aligned}
(\partial_t - \Delta)v = -\Big(&v^3 + 3\sum_{\tau\in\mathcal{W}} (-1)^{\frac{m(\tau)-1}{2}} v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(\tau) \\
&+ 3\sum_{\tau_1,\tau_2\in\mathcal{W}} (-1)^{\frac{m(\tau_1)+m(\tau_2)-2}{2}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(\tau_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(\tau_2) \\
&- \sum_{\substack{\tau_1,\tau_2,\tau_3\in\mathcal{W} \\ |\tau_1|+|\tau_2|+|\tau_3|>-8}} (-1)^{\frac{m(\tau_1)+m(\tau_2)+m(\tau_3)-3}{2}} \mathbb{X}_\bullet \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)\Big).
\end{aligned}
\tag{4.7.7}
$$

Here, the tree expansion that governs v is given by

$$V(z) = \sum_{\tau\in\mathcal{N}} \Upsilon_z(\tau)\mathcal{I}(\tau)$$

where Υ_z is defined as in Definition 4.6.6 using the function $v(z)$ and the parameters $v_{\mathbf{1}} = v$ and $v_{\mathbf{X}} = \mathcal{D}^{\mathbb{X}}v$.

The following statement is then straightforward.

Lemma 4.7.5. *There is a one to one correspondence between solutions in the sense of Definition 4.7.3 to those of Definition 4.7.4, the correspondence is given by taking ϕ which is a solution in the sense Definition 4.7.3 and mapping it to $v = v_{\mathbf{1}}$ which will be a solution in the sense of* (4.7.6)*.*

4.8 A useful class of local products

This section is somewhat orthogonal to the main result of this chapter. Here we present a particular subset of local products, lifts of a smooth noise ξ, which are defined in terms of recursive procedure that guarantees that the renormalised product appearing in (4.7.5) is a local polynomial in ϕ and its derivatives. This class of local products also includes those that satisfy the necessary uniform stochastic estimates in order to go to the rough setting, namely the BPHZ renormalisation of [10, 18].

4.8.1 Another derivative edge

Our class of local products will, for δ sufficiently small, allow the renormalised product $\phi^{\circ_{\mathbb{X}}3}$ to involve spatial derivatives $\partial_i\phi$ for $1 \le i \le d$.

To describe the generation of these derivatives in terms of operations on trees we will introduce yet another set of edges $\{\mathcal{I}_i^-\}_{i=1}^d$ and another set of planted trees

$$\mathcal{T}_{l,-} = \mathcal{T}_l \cup \big\{\mathcal{I}_i^-(\tau) : \tau \in \mathcal{T}_r, 1 \le i \le d\big\} \cup \{\mathcal{I}_i^+(\mathbf{X}_i) : 1 \le i \le d\} .$$

We also adopt the notational convention that

$$\mathcal{I}_i^-(\mathbf{X}_i) = \mathcal{I}_i^+(\mathbf{X}_i),\ \mathcal{I}_i^-(\mathbf{1}) = 0,\ \text{and}\ \mathcal{I}_i^-(\mathbf{X}_j) = 0\ \text{if}\ i \neq j\ .$$

Both the new set of edges $\{\mathcal{I}_i^-\}_{i=1}^d$ and the set of edges $\{\mathcal{I}_i^+\}_{i=1}^d$ introduced in Section 4.4 should be thought of as representing a spatial derivative of a solution to a heat equation. However these two sets of edges play different roles in our argument: a symbol $\mathcal{I}_i^+(\tau)$ for $\tau \in \widetilde{\mathcal{N}}$ is used to describe centring terms while the terms $\mathcal{I}_i^-(\widetilde{\tau})$ for $\widetilde{\tau} \in \mathcal{T}_r$ are only used so we can write tree expansions for derivatives generated by our renormalised products.

In particular, since the renormalised product associated to a local product $\mathbb{X}$ is defined in terms of the path built from it, it will be useful to extend this path to act on such $\mathcal{I}_i^-(\widetilde{\tau})$ trees and the natural action to choose here will be different than the action of the path on $\mathcal{I}_i^+(\tau)$ trees.

Another difference between these two sets of derivative edges is that while we adopted the convention that, for any $\tau \in \mathcal{T}_r \setminus \widetilde{\mathcal{N}}$, one has $\mathcal{I}_i^+(\tau) = 0$. We do not adopt the same convention for $\mathcal{I}_i^-(\tau)$.

For convenience we will treat the symbols $\mathcal{I}_i^-(\mathbf{X}_i)$ and $\mathcal{I}_i^+(\mathbf{X}_i)$ as the same and also adopt the convention that $\mathcal{I}_i^-(\mathbf{1}) = 0$. We also extend our notion of order to $\mathcal{T}_{l,-}$ by setting, for $\tau \in \mathcal{T}_r$, $|\mathcal{I}_i^-(\tau)| = |\tau| + 1$.

We extend any local product $\mathbb{X}$ to the new trees we have added in $\mathcal{T}_{l,-}$ by setting, for $1 \leq i \leq d$,

$$\mathbb{X}_z \mathcal{I}_i^-(\tau) = \begin{cases} \partial_i(\mathcal{L}^{-1}\mathbb{X}_\bullet\tau)(z) & \text{if } \tau \in \mathcal{T}_r\ , \\ 1 & \text{if } \tau = \mathbf{X}_i\ . \end{cases} \tag{4.8.1}$$

4.8.2 Operations on $\mathcal{I}_i^-$ trees

Given a local product $\mathbb{X}$, we extend the corresponding path to $\mathcal{I}_i^-$ trees by setting, for any $\tau \in \mathcal{T}_r$ and $1 \leq i \leq d$,

$$\mathbb{X}_{z,w}\mathcal{I}_i^-(\tau) = \big(\mathbb{X}_z \otimes \mathbb{X}_w^{\mathrm{rec}}\big)\Delta\mathcal{I}_i^-(\tau)\ . \tag{4.8.2}$$

where we extend the formulae of (4.4.3) by setting, for $1 \leq i \leq d$,

$$\Delta\mathcal{I}_i^-(\tau) = (\mathcal{I}_i^- \otimes \mathrm{Id})\Delta\tau + \mathcal{I}_i^+(\mathbf{X}_i) \otimes \mathcal{I}_i^+(\tau),\ \tau \in \mathcal{T}_r\ .$$

Remark 4.8.1. We remark that our convention that $\mathcal{I}_i^-(\mathbf{X}_i) = \mathcal{I}_i^+(\mathbf{X}_i)$ also seems natural since this guarantees (4.4.5) holds for σ of the form $\mathcal{I}_i^-(\tau)$, but this observation

will not play any role in our argument.

We then have the following easy lemma.

Lemma 4.8.2. *Let $\mathbb{X}$ be a local product, then for any $1 \le i \le d$ and $\tau \in W w \sqcup \mathcal{N}$, one has*

$$\partial_i \mathbb{X}_{z,w} \mathcal{I}(\tau) = \mathbb{X}_{z,w} \mathcal{I}_i^-(\tau) \text{ for any } \tau \in \mathcal{W} \cup \mathcal{N} \,, \tag{4.8.3}$$

where the derivative ∂_i above acts in the variable z. In particular, one has

$$\mathbb{X}_{z,z} \mathcal{I}_i^-(\tau) = 0 \text{ if } |\mathcal{I}_i^-(\tau)| > 0 \tag{4.8.4}$$

.

Moreover, if $\Theta : \mathcal{N} \to C^\infty$ has the property that, for some $1 < \gamma < 2$, we have $[U^{\mathbf{1}}]_\gamma < \infty$ (where $U^{\mathbf{1}}_{\gamma-2}$ is defined as in (4.6.5)*) then we have that*

$$\partial_i \Theta_z(\mathbf{1}) = \sum_{\substack{\tau \in \mathcal{N} \\ |\tau| \le -1}} \Theta_z(\tau) \mathbb{X}_{z,z} \mathcal{I}_i^-(\tau) \,. \tag{4.8.5}$$

Proof. The first statement (4.8.3) is a straightforward computation using (4.8.3) and (4.8.1) and (4.8.2). The second statement (4.8.4) then follows from the first one and the order bound $[\mathbb{X} : \mathcal{I}(\tau)]$ for such τ.

Finally, the third statement (4.8.5) follows immediately from combining (4.8.3) and Lemma 4.6.3. □

4.8.3 A recipe for local products

With this notation in hand, our recipe for building a local product $\mathbb{X}$ will be to first specify the smooth function $\mathbb{X}_z \Xi$ and then inductively define, for $\tau \in \mathcal{Q}$ (recall that $\mathcal{Q}$ was defined in Definition 4.3.6),

$$\mathbb{X}_z \tau = \mathbb{X}_z R \tau \tag{4.8.6}$$

where $R : \mathcal{Q} \to \mathrm{Alg}(\mathcal{T}_{l,-})$, and on the right-hand side, we apply $\mathbb{X}_z$ multiplicatively over the planted trees appearing in the forests of $\mathrm{Alg}(\mathcal{T}_{l,-})$ and use the conventions of Section 4.3.3 and (4.8.2) to reduce the right-hand side to evaluating $\mathbb{X}_z$ on $\mathcal{N} \cup \mathcal{W}$. For this to be a well-defined way to construct local products the map R must satisfy the following two criteria

- For the induction (4.8.6) to be closed, it is natural to enforce that R should have a triangular structure in that, for any $\tau \in \mathcal{Q}$, any planted tree appearing in a forest appearing in $R\tau$ should be of the form $\mathcal{I}(\widetilde{\tau})$ or $\mathcal{I}_i^-(\widetilde{\tau})$ with $\widetilde{\tau}$ strictly fewer edges

than τ.

- In order for $\mathbb{X}_z$ to be invariant under permutations of non-commutative tree products it also natural to enforce that R act covariantly with respect to such permutations.

If we have a map R as above and use it to build a local product $\mathbb{X}$ using (4.8.6) then we say $\mathbb{X}$ is built from R.

For what follows it is useful to define the map q_F which takes tree products to forest products, namely q_F maps $\mathring{\mathcal{N}} \cup \mathring{\mathcal{W}} \ni \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \mapsto \mathcal{I}(\tau_1)\cdot\mathcal{I}(\tau_2)\cdot\mathcal{I}(\tau_3) \in \mathrm{Alg}(\mathcal{T}_l)$.

Remark 4.8.3. One possible choice for a renormalisation operator R_{mult} is setting, for each $\tau \in \mathcal{Q}$, $R_{\mathrm{mult}}\tau = q_F\tau$. If one uses R_{mult} to build a local product $\mathbb{X}$ then it follows that $\mathbb{X}$ is a multiplicative local product.

However, in order to allow more flexibility than a multiplicative local product but still make it easy to show that that the product $\phi^{\circ_{\mathbb{X}}3}$ in (4.7.5) admits a nice formula, we impose a structural assumption on the operator R.

This assumption can be expressed in terms of a slightly modified version of our earlier defined coproduct.

4.8.4 A modified coproduct and local renormalisation operators

The modified coproduct is defined with a map C_-, modification of C_+. $C_- : (\overline{\tau}, \tau) \in \mathcal{T} \times \mathcal{T} \to \mathcal{F}$ is given by the table below.

$\tau \setminus \overline{\tau}$	$\mathbf{1}$	$\mathbf{X}_i$	Ξ	$\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(\overline{\tau}_3)$
$\mathbf{1}$	$\mathcal{I}(\mathbf{1})$	0	0	0
$\mathbf{X}_j$	$\mathcal{I}(\mathbf{X}_j)$	$\mathcal{I}_i^-(\mathbf{X}_j)$	0	0
Ξ	$\mathcal{I}(\Xi)$	$\mathcal{I}_i^-(\Xi)$	1	0
$\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$	$\mathcal{I}(\tau)$	$\mathcal{I}_i^-(\tau)$	0	$C_-(\overline{\tau}_1, \tau_1)C_-(\overline{\tau}_2, \tau_2)C_-(\overline{\tau}_3, \tau_3)$

Table 4.2: This table gives a recursive definition of $C_-(\overline{\tau}, \tau)$. Possible values of τ are displayed in the first column, while possible values of $\overline{\tau}$ are shown in the first row. The corresponding values of $C_-(\overline{\tau}, \tau)$ are shown in the remaining fields.

The difference between C_+ and C_- is the removal of the projection on positive planted trees. Similarly, we never assume that $\mathcal{I}_i^-(\tau) = 0$ if $\tau \notin \widetilde{\mathcal{N}}$. C_- also satisfies Lemma 4.4.12, with the first implication in (4.4.6) being an equivalence in this case.

The following lemma, which follows in an immediate way from the definition of our sets of trees $\mathcal{N}$ and $\mathcal{W}$, will be useful when we try to drive an explicit formula for $\phi^{\circ_{\mathbb{X}}3}$.

Lemma 4.8.4. *For any fixed $\overline{\tau} \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$,*

$$p_{\leqslant 0} \sum_{\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} C_-(\overline{\tau}, \tau)$$
$$= \begin{cases} \big(\sum_{\tau \in \mathcal{W}} \mathcal{I}(\tau) + \mathcal{I}(\mathbf{1})\big) & \text{if } m_{\mathbf{1}}(\overline{\tau}) = 1 \ , \\ \big(\sum_{\tau \in \mathcal{W}} \mathcal{I}(\tau) + \mathcal{I}(\mathbf{1})\big) \cdot \big(\sum_{\tau \in \mathcal{W}} \mathcal{I}(\tau) + \mathcal{I}(\mathbf{1})\big) & \text{if } m_{\mathbf{1}}(\overline{\tau}) = 2 \ , \\ \sum_{\substack{\tau \in \mathcal{N} \cup \mathcal{W} \\ |\tau| \leq -1}} \mathcal{I}_i^-(\tau) & \text{if } m_{\mathbf{X}_i}(\overline{\tau}) = 1 \ . \end{cases}$$

Above, $p_{\leqslant} : \mathrm{Alg}(\mathcal{T}_{l,-}) \to \mathrm{Alg}(\mathcal{T}_{l,-})$ is the projection that annihilates any forest of planted trees that contains a planted tree of strictly positive degree.

Definition 4.8.5. *Given a map $r : \mathcal{Q} \to \mathbb{R}$ such that r is invariant under permutations we define a corresponding map $R : \mathcal{Q} \to \mathrm{Alg}(\mathcal{T}_{l,-})$ by defining*

$$R(\tau) = q_F \tau + \sum_{\tau' \in \mathcal{Q}} r(\tau') C_-(\tau', \tau) \ . \tag{4.8.7}$$

Note that counterterm maps r and local renormalisation operators R determine each other uniquely.

We then immediately have the following lemma.

Lemma 4.8.6. *Let R be a local renormalisation operator. .*

Given a local product $\mathbb{X}$ built from R, the formula (4.8.6) *defining $\mathbb{X}$ for $\tau \in \mathcal{Q}$ actually also holds as an identity for $\tau \in \mathring{\mathcal{N}} \setminus \mathcal{Q}$ where R is itself is extended to $\mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$ by applying the formula* (4.8.7)*.*

Suppose we are given a $\mathbb{X}$ built from R. Then the formula (4.8.6) allows us to compute the action of $\mathbb{X}_\bullet$ on any element $\tau \in \mathcal{N} \cup \mathcal{W}$ in terms of its actions on simpler trees. However, the starting formula for $\phi^{\circ_{\mathbb{X}} 3}$ involves the action of the corresponding path $\mathbb{X}_{z,z}$. Therefore in order to work out an explicit formula for $\phi^{\circ_{\mathbb{X}} 3}$ it would be good to have an analogue of (4.8.6) for paths $\mathbb{X}_{\bullet,\bullet}$ instead of just the underlying local product $\mathbb{X}_\bullet$. Heuristically the idea for getting such a formula is showing that the action of a local renormalisation operator will "commute" with our centring operations. To this end we have the following lemma.

Lemma 4.8.7. *Let R be a local renormalisation operator. Then one has, for any $\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$, the identity*

$$\Delta R \tau = (R \otimes \mathrm{Id}) \Delta \tau \ . \tag{4.8.8}$$

Proof.

$$\Delta R(\tau) = \Delta q_F \tau + \sum_{\overline{\overline{\tau}} \in \mathcal{T}} r(\overline{\overline{\tau}}) \Delta C_-(\overline{\overline{\tau}}, \tau).$$

We split the sum above depending on $m_{\mathbf{1}}(\overline{\overline{\tau}})$ and $m_{\mathbf{x}}(\overline{\overline{\tau}})$, for $\overline{\overline{\tau}} \leqslant \tau$. If $m_{\mathbf{1}}(\overline{\overline{\tau}}) = m_{\mathbf{x}}(\overline{\overline{\tau}}) = 0$, then $C_-(\overline{\overline{\tau}}, \tau) = \delta_{\{\tau = \overline{\overline{\tau}}\}}$.

If $m_{\mathbf{1}}(\overline{\overline{\tau}}) = 1$, then we have $C_-(\overline{\overline{\tau}}, \tau) = \mathcal{I}(\sigma)$ where $\sigma \subset \tau$ and

$$\Delta \mathcal{I}(\sigma) = \sum_{\widetilde{\tau} \in \mathring{\mathcal{N}} \cup \mathcal{W}} \mathcal{I}(\widetilde{\tau}) \otimes C_+(\widetilde{\tau}, \widetilde{\sigma}).$$

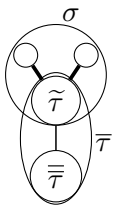
For each element $\widetilde{\tau}$ in this sum, we define a corresponding $\overline{\tau}$ by replacing the occurrence of σ in τ identified above by $\widetilde{\tau}$. We have $\overline{\overline{\tau}} \leqslant \overline{\tau} \leqslant \tau$ and by the inductive formulas,

$$\mathcal{I}(\widetilde{\tau}) = C_-(\overline{\overline{\tau}}, \overline{\tau})$$

and

$$C_+(\widetilde{\tau}, \widetilde{\sigma}) = C_+(\overline{\tau}, \tau).$$

The following picture is a representation of τ and the relation between its different subtrees, for one choice of $\widetilde{\tau}$, to give an intuition of the proof. Drawn above the thicker lines is $C_+(\widetilde{\tau}, \widetilde{\sigma}) = C_+(\overline{\tau}, \tau)$, which in this example is a product of two planted trees.



If $m_{\mathbf{x}}(\overline{\overline{\tau}}) = 1$, the same holds by replacing $\mathcal{I}$ by $\mathcal{I}_i^-$ in the argument.

If $m_{\mathbf{1}}(\overline{\overline{\tau}}) = 2$, then we write $C_-(\overline{\overline{\tau}}, \tau) = \mathcal{I}(\sigma_1) \cdot \mathcal{I}(\sigma_2)$ where $\sigma_i \subset \tau$ and for $i = 1, 2$,

$$\Delta \mathcal{I}(\sigma_i) = \sum_{\widetilde{\tau}_i \in \mathring{\mathcal{N}} \cup \mathcal{W}} \mathcal{I}(\widetilde{\tau}_i) \otimes C_+(\widetilde{\tau}_i, \widetilde{\sigma}_i).$$

For each element $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$, we define $\overline{\tau}$ by replacing σ_i by $\widetilde{\tau}_i$ in τ. We have $\overline{\overline{\tau}} \leqslant \overline{\tau} \leqslant \tau$ and by the inductive formulas,

$$\mathcal{I}(\widetilde{\tau}_1) \cdot \mathcal{I}(\widetilde{\tau}_2) = C_-(\overline{\overline{\tau}}, \overline{\tau})$$

and

$$C_+(\widetilde{\tau}_1, \widetilde{\sigma}_1) \cdot C_+(\widetilde{\tau}_2, \widetilde{\sigma}_2) = C_+(\overline{\tau}, \tau).$$

In all the cases discussed, we can index the sum induced by the coproduct in terms of $\overline{\tau}$ instead of $\widetilde{\tau}$ or $\widetilde{\tau}_1, \widetilde{\tau}_2$. Permutation of that sum with the sum over $\overline{\overline{\tau}}$ then gives:

$$\begin{aligned}\Delta R(\tau) =&\Delta q_F \tau + \sum_{\overline{\tau} \in \mathring{\mathcal{N}} \cup \mathcal{W}} (\sum_{\overline{\overline{\tau}} \in \mathcal{T}} r(\overline{\overline{\tau}}) C_-(\overline{\overline{\tau}}, \overline{\tau})) \otimes C_+(\overline{\tau}, \tau) \\ =& \sum_{\overline{\tau} \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} (q_F \overline{\tau} + \sum_{\overline{\overline{\tau}} \in \mathcal{T}} r(\overline{\overline{\tau}}) C_-(\overline{\overline{\tau}}, \overline{\tau})) \otimes C_+(\overline{\tau}, \tau) \\ =& \sum_{\overline{\tau} \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} R(\overline{\tau}) \otimes C_+(\overline{\tau}, \tau) \\ =&(R \otimes \mathrm{Id}) \Delta \tau \ ,\end{aligned}$$

where above we adopt the convention that $r(\tau') = 0$ if $\tau' \notin \mathcal{Q}$. □

With this identity we can now give an analog of (4.8.6) for our paths.

Lemma 4.8.8. *Suppose that the local product $\mathbb{X}$ was built from an local renormalisation map R. Then, for any $x, y \in \mathbb{R}^d$, and tree $\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}$ one has*

$$\mathbb{X}_{x,y} \tau = \mathbb{X}_{x,y} R \tau \tag{4.8.9}$$

where on the right-hand side we extend $\mathbb{X}_{x,y}$ to forests of planted trees multiplicatively.

Proof. Our proof is by induction in the size of τ. The bases cases where $m_\tau = 3$ are straightforward to check by hand. For the inductive step, we note that one has

$$\mathbb{X}_{x,y} \tau = (\mathbb{X}_x \otimes \mathbb{X}_y^{\mathrm{rec}}) \Delta \tau = (\mathbb{X}_x R \otimes \mathbb{X}_y^{\mathrm{rec}}) \Delta \tau = (\mathbb{X}_x \otimes \mathbb{X}_y^{\mathrm{rec}}) \Delta R \tau = \mathbb{X}_{x,y} R \tau \ ,$$

where in the second equality we used Lemma 4.8.6 and in the third equality we used Lemma 4.8.7. □

Remark 4.8.9. We describe how the renormalisation of Φ^4_3 (which in our setting corresponds to fixing $\delta = 1/2-$ with Gaussian noise) used in previous works such as [56] corresponds to a choice of a local renormalisation operator R.

We define $\mathcal{Q}_{\text{wick}}$ to be the three different elements of $\mathring{\mathcal{N}}$ obtained by permuting the tree product in $\mathcal{I}(\mathbf{1})\mathcal{I}(\Xi)^2$, that is

$$\mathcal{Q}_{\text{wick}} = \{\mathcal{I}(\mathbf{1})\mathcal{I}(\Xi)^2, \mathcal{I}(\Xi)\mathcal{I}(\mathbf{1})\mathcal{I}(\Xi), \mathcal{I}(\Xi)^2\mathcal{I}(\mathbf{1})\}$$

and similarly define $\mathcal{Q}_{\text{sunset}}$ to be the collection of the 9 different elements of $\mathring{\mathcal{N}}$ which

are obtained by permutations of the tree product in

$$\mathcal{I}(\Xi)\mathcal{I}\big(\mathcal{I}(\Xi)\mathcal{I}(\mathbf{1})\mathcal{I}(\Xi)\big)\mathcal{I}(\Xi) = \overset{0}{\bullet}\!\!\Big\downarrow\!\!\bullet\!\!\Big\downarrow\!\!\bullet\,.$$

There are nine elements because there are three different orders for each of the two tree products appearing in this tree, for instance one also has

$$\mathcal{I}\big(\mathcal{I}(\mathbf{1})\mathcal{I}(\Xi)^2\big)\mathcal{I}(\Xi)^2 \in \mathcal{Q}_{\text{sunset}}\ .$$

The corresponding counterterm map r is given by

$$r(\tau) = \begin{cases} -C_{\text{wick}} & \text{if } \tau \in \mathcal{Q}_{\text{wick}}, \\ -C_{\text{sunset}} & \text{if } \tau \in \mathcal{Q}_{\text{sunset}}, \\ 0 & \text{otherwise.} \end{cases}$$

where one has

$$\begin{aligned} C_{\text{wick}} =& \mathbb{E}[(\mathcal{L}^{-1}\xi)(0)^2]. \\ C_{\text{sunset}} =& \mathbb{E}[\theta(0)\mathcal{L}^{-1}\theta(0)], \end{aligned}$$

where ξ is our (regularised) noise, P is the space-time Green's function for the heat kernel and θ is defined by

$$\theta(z) = (\mathcal{L}^{-1}\xi)(z)^2 - \mathbb{E}[(\mathcal{L}^{-1}\xi)(0)^2] = (\mathcal{L}^{-1}\xi)(z)^2 - C_{\text{wick}}.$$

The promised local renormalisation operator is then given by building R from r as in (4.8.7).

As an example, we compute

$$R\,\Big\Downarrow\!\!\bullet = \Big\uparrow\Big\uparrow\Big\uparrow - 3C_{\text{wick}}\Big\uparrow \quad \text{and} \quad R\,\overset{0}{\bullet}\!\!\Big\downarrow\!\!\bullet\!\!\Big\downarrow\!\!\bullet = \Big\uparrow\,\overset{0}{\bullet}\!\!\Big\downarrow\!\!\bullet\!\!\Big\uparrow - C_{\text{wick}}\,\overset{0}{\bullet}\!\!\Big\downarrow\!\!\bullet - C_{\text{sunset}}\,\overset{0}{\bullet}\!\!\Big|\,.$$

4.8.5 Formula for the renormalised cube

The next proposition gives the explicit formulae for our renormalised product that promised at the beginning of this section.

Proposition 4.8.10. *Let $\mathbb{X}$ be built from a local renormalisation operator R.*

Fix smooth functions $v_{\mathbf{1}}, v_{\mathbf{X}_1}, \dots, v_{\mathbf{X}_d} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and let

$$\Phi(z) = \sum_{\tau \in \mathcal{N} \cup \mathcal{W}} \Upsilon_z(\tau) \mathcal{I}(\tau)$$

where Υ is defined in terms of the parameters $v_{\mathbf{1}}$ and $v_{\mathbf{X}}$, and

$$\phi(z) = \mathbb{X}_{z,z}\Phi(z) = \Upsilon_z(\mathbf{1}) + \sum_{\tau \in \mathcal{W}} \Upsilon(\tau) \mathbb{X}_z \mathcal{I}(\tau) \ .$$

Moreover, suppose that, for some $1 < \gamma < 2$, if $U^{\mathbf{1}}_{\gamma-2}$ as in (4.6.5) *with $\Theta_\bullet(\cdot) = \Upsilon_\bullet(\cdot)|_{\mathcal{N}}$, we have that $[U^{\mathbf{1}}]_\gamma < \infty$.*

Then, if we define $\phi^{\circ \mathbb{X}^3}$ as in Definition 4.7.1 using Θ as our tree expansion for ϕ, we then have

$$\phi^{\circ \mathbb{X}^3}(z) = \phi^3(z) - r_1 - r_\Phi \phi(z) - r_{\Phi^2}\phi^2(z) - \sum_{i=1}^{d} r_{\partial_i \Phi} \partial_i \phi(z), \tag{4.8.10}$$

where

$$\begin{aligned} r_1 &= \sum_{\substack{\overline{\tau} \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}, \\ m_{\mathbf{1}}(\overline{\tau}) + m_{\mathbf{X}}(\overline{\tau}) = 0}} (-1)^{\frac{m(\overline{\tau})-1}{2}} r(\overline{\tau}), \quad r_\Phi = \sum_{\substack{\overline{\tau} \in \mathring{\mathcal{N}} \\ m_{\mathbf{1}}(\overline{\tau}) = 1}} (-1)^{\frac{m(\overline{\tau})-1}{2}} r(\overline{\tau}), \\ r_{\Phi^2} &= \sum_{\substack{\overline{\tau} \in \mathring{\mathcal{N}} \\ m_{\mathbf{1}}(\overline{\tau}) = 2}} (-1)^{\frac{m(\overline{\tau})-1}{2}} r(\overline{\tau}), \quad r_{\partial_i \Phi} = \sum_{\substack{m_{\mathbf{X}_i}(\overline{\tau}) = 1 \\ \overline{\tau} \in \mathring{\mathcal{N}}}} (-1)^{\frac{m(\overline{\tau})-1}{2}} r(\overline{\tau}). \end{aligned} \tag{4.8.11}$$

Here $r : \mathcal{Q} \to \mathbb{R}$ is the map from which R is built.

Proof. We have

$$\begin{aligned} \phi^{\circ \mathbb{X}^3}(z) &= \sum_{\tau_1, \tau_2, \tau_3 \in \mathcal{N} \cup \mathring{\mathcal{W}}} \Upsilon_z(\tau_1)\Upsilon_z(\tau_2)\Upsilon_z(\tau_3)\mathbb{X}_{z,z}\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3) \\ &= \sum_{\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} \Upsilon_z(\tau)\mathbb{X}_{z,z}\tau \\ &= \sum_{\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} \Upsilon_z(\tau)\mathbb{X}_{z,z}R\tau \\ &= \sum_{\tau \in \mathring{\mathcal{N}} \cup \mathring{\mathcal{W}}} \Upsilon_z(\tau)(\mathbb{X}_{z,z}q_F\tau + \sum_{\overline{\tau} \in \mathcal{T}} r(\overline{\tau})\mathbb{X}_{z,z}C_-(\overline{\tau}, \tau)) \ , \end{aligned}$$

where the first equality follows from the definition of renormalised local products, the second equality comes from Lemma 4.6.9, and the third equality comes from Lemma 4.8.8.

By appealing to Lemma 4.6.9 once more, we can rewrite the first term of the last line above as

$$-\mathbb{X}_{z,z}q_F\sum_{\tau\in\mathring{\mathcal{N}}\cup\mathring{\mathcal{W}}}\Upsilon_z(\tau)\tau=\mathbb{X}_{z,z}q_F\big(\sum_{\tau\in\mathcal{N}\cup\mathcal{W}}\Upsilon_z(\tau)\mathcal{I}(\tau)\big)^3$$
$$=\big(\sum_{\tau\in\mathcal{N}\cup\mathcal{W}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau)\big)^3=\phi^3(z)\,.$$

By using Lemma 4.8.4 (note that $\mathbb{X}_{z,z}p_{\leq 0}=\mathbb{X}_{z,z}$ on $\mathrm{Alg}(\mathcal{T}_{l,-})$) followed by equation (4.6.20), and using the fact that $\mathbb{X}_{z,z}\sigma=0$ if $|\sigma|>0$, we have

$$\sum_{\tau\in\mathring{\mathcal{N}}\cup\mathring{\mathcal{W}}}\Upsilon_z(\tau)\sum_{\overline{\tau}\in\mathring{\mathcal{N}}\cup\mathring{\mathcal{W}}}r(\overline{\tau})\mathbb{X}_{z,z}C_-(\overline{\tau},\tau) \tag{4.8.12}$$
$$=\sum_{\substack{\overline{\tau}\in\mathring{\mathcal{N}}\cup\mathring{\mathcal{W}},\\ m_{\mathbf{1}}(\overline{\tau})+m_{\mathbf{x}}(\overline{\tau})=0}}(-1)^{\frac{m(\overline{\tau})-1}{2}}r(\overline{\tau})1$$
$$+\sum_{\substack{\overline{\tau}\in\mathring{\mathcal{N}}\\ m_{\mathbf{1}}(\overline{\tau})=1}}(-1)^{\frac{m(\overline{\tau})-1}{2}}r(\overline{\tau})\sum_{\tau\in\{\mathbf{1}\}\cup\mathcal{W}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau)$$
$$+\sum_{\substack{\overline{\tau}\in\mathring{\mathcal{N}}\\ m_{\mathbf{1}}(\overline{\tau})=2}}(-1)^{\frac{m(\overline{\tau})-1}{2}}r(\overline{\tau})\big(\sum_{\tau\in\{\mathbf{1}\}\cup\mathcal{W}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau)\big)^2$$
$$+\sum_{i=1}^{d}\sum_{\substack{\overline{\tau}\in\mathring{\mathcal{N}}\\ m_{\mathbf{x}_i}(\overline{\tau})=1}}(-1)^{\frac{m(\overline{\tau})-1}{2}}r(\overline{\tau})\sum_{\substack{\tau\in\mathcal{N}\cup\mathcal{W}\\ |\tau|\leq -1}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}_i^-(\tau).$$

We then obtain the desired result by observing that, for the second and third terms on the right-hand side above,

$$\sum_{\tau\in\{\mathbf{1}\}\cup\mathcal{W}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}(\tau)=\Upsilon_z(\mathbf{1})+\sum_{\tau\in\mathcal{W}}\Upsilon_z\mathbb{X}_z\mathcal{I}(\tau)=\phi(z)$$

and, for the third term on the right-hand side above, we have, for $1\leq i\leq d$,

$$\sum_{\substack{\tau\in\mathcal{N}\cup\mathcal{W}\\ |\tau|\leq -1}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}_i^-(\tau)=\sum_{\tau\in\mathcal{W}}\Upsilon(\tau)\mathbb{X}_z\mathcal{I}_i^-(\tau)+\sum_{\substack{\tau\in\mathcal{N}\\ |\tau|\leq -1}}\Upsilon_z(\tau)\mathbb{X}_{z,z}\mathcal{I}_i^-(\tau)$$
$$=\sum_{\tau\in\mathcal{W}}\Upsilon_z(\tau)\partial_i\mathbb{X}_z\mathcal{I}(\tau)+\partial_i\Upsilon_z(\mathbf{1})=\partial_i\phi(z)\,.$$

For the first equality of the second line above we used Lemma 4.8.2 - in particular, (4.8.5) - with $\Theta_\bullet(\cdot)=\Upsilon_\bullet(\cdot)|_{\mathcal{N}}$. □

Remark 4.8.11. Under the assumptions of Proposition 4.8.10 one can also show that

each one of the renormalised products in (4.7.7) can also be expressed in terms of local polynomials of $v(z)$, $\{\partial_i v(z)\}_{i=1}^d$, and $\{\mathbb{X}_z\mathcal{I}(\tau) : \tau \in \mathcal{W}\}$.

However, we refrain from doing this because the index sets for the summations that define the analogs of the constants (4.8.11) become quite complicated.

Remark 4.8.12. Returning to the example of Φ^4_3 described in Remark 4.8.9, one then sees that $r_{\partial_i\Phi} = r_{\Phi^2} = r_1 = 0$ for all $1 \le i \le d$ and

$$r_\Phi = \sum_{\tau\in\mathcal{Q}_{\text{wick}}} (-1)^{\frac{3-1}{2}}(-C_{\text{wick}}) + \sum_{\tau\in\mathcal{Q}_{\text{sunset}}} (-1)^{\frac{5-1}{2}}(-C_{\text{sunset}}) = 3C_{\text{wick}} - 9C_{\text{sunset}} \,.$$

4.9 Main result

4.9.1 Statement of main theorem

The main theorem of this chapter is a generalisation of Theorem 3.2.1 to the full subcritical regime. It allows therefore to treat cases with any number of trees appearing in the expansion of the solution. In this chapter, we are therefore able to generalise a result where renormalisation was treated by hand, to the (modified) setting of regularity structures.

We recall the definition of the parabolic cylinders D and D_R:

$$D = (0,1) \times \{|x| < 1\}, \quad D_R = (R^2, 1) \times \{|x| < 1 - R\},$$

and of the parabolic ball of centre $z = (t, x)$ and radius R in this metric d, looking only into the past:

$$B(z, R) = \{\overline{z} = (\overline{t}, \overline{x}) \in \mathbb{R} \times \mathbb{R}^d, d(z, \overline{z}) < R, \overline{t} < t\}. \tag{4.9.1}$$

Theorem 4.9.1. *There exists a constant C such that if v is a pointwise solution on D to the remainder equation driven by a local product $\mathbb{X}$, according to Definition 4.7.4 then*

$$\|v\|_{D_R} \leqslant C \max\Big\{\frac{1}{R}, [\mathbb{X}; \tau]^{\frac{1}{\delta m_\Xi(\tau)}}, |\tau| < 0, m_\Xi(\tau) \neq 0\Big\}. \tag{4.9.2}$$

This theorem generalises to an arbitrary domain $\widetilde{D}$ in the following way: the local path is defined in a similar way, only replacing the cutoff function ρ by a cutoff function that has value 1 on a 1-enlargement of $\widetilde{D}$, and vanishes on a 2 enlargement of the set. Then for every point in $\widetilde{D}$, one can obtain a bound depending only on the path by applying a translated version of the theorem, for $R = 1$.

The following corollary is a reformulation of this theorem following from Defini-

tion 4.7.3.

Corollary 4.9.2. *There exists a constant C such that if ϕ is a pointwise solution to equation* (4.7.5) *driven by a local product $\mathbb{X}$ then for $v = \phi - \sum_{w\in\mathcal{W}} \Upsilon(w)\mathbb{X}\mathcal{I}(w)$, the bound* (4.9.2) *holds.*

The following result is a particular case of the local product introduced in Section 4.8. It follows from Proposition 4.8.10.

Corollary 4.9.3. *Let R be a local renormalisation operator, and $\mathbb{X}$ be the local product built from R. There exists a constant C such that if ϕ is a pointwise solution to*

$$(\partial_t - \Delta)\phi = -\phi^3(z) + r_1 + r_\Phi\phi(z) + r_{\Phi^2}\phi^2(z) + \sum_{i=1}^{d} r_{\partial_i\Phi}\partial_i\phi(z), \tag{4.9.3}$$

where the coefficients $r_1,\ r_\Phi,\ r_{\Phi^2}$ and r_{∂_i} are given by (4.8.11)*, then for $v = \phi - \sum_{w\in\mathcal{W}} \Upsilon(w)\mathbb{X}\mathcal{I}(w)$, the bound* (4.9.2) *holds.*

The typical application of these results concerns paths $\mathbb{X}$ which are constructed from a Gaussian noise ξ. In this case, one typically has that for a given tree τ the quantity $[\mathbb{X};\tau]$ is in the (inhomogeneous) Wiener chaos of order $m_\Xi(\tau)$ (see [27]). In particular, in this case one gets for some $\lambda > 0$,

$$\mathbb{E}[\exp(\lambda[\mathbb{X};\tau]^{\frac{2}{m_\Xi(\tau)}})] < \infty.$$

This implies the following corollary:

Corollary 4.9.4. *If ξ is a Gaussian noise and the path $\mathbb{X}$ is built from the local renormalisation operator, if v is a pointwise solution to the remainder equation driven by a local product $\mathbb{X}$ on D, according to Definition 4.7.4 then there exists a constant $\overline{\lambda}$ such that*

$$\mathbb{E}[\exp(\overline{\lambda}\|v\|^{2\delta}_{D_{\frac{1}{2}}})] < \infty.$$

Remark 4.9.5. The results presented here also imply a bound for the corresponding elliptic equation in dimension $6-$, i.e.

$$\Delta\phi = \phi^3 - \xi \qquad\qquad x \in \mathbb{R}^6,$$

where ξ is a 6-dimensional white noise which is slightly regularized (e.g. by applying $(1-\Delta)^{-\overline{\delta}}$ for an arbitrary $\overline{\delta} > 0$). The four and five dimensional versions of this equation were recently studied in [4, 35]. Our Corollary 4.9.4 can be applied directly, if ϕ is viewed as a stationary solution of the parabolic equation (i.e. with $\partial_t\phi = 0$). However, to treat the elliptic case it would be more natural to define the action of $\mathbb{X}$ on $\mathcal{I}$

slightly differently in terms of the inverse Laplace, rather than the inverse heat operator. Such a change could be implemented easily.

4.9.2 Proof of Theorem 4.9.1

We remind the reader that the notation τ always refers to an unplanted tree. In particular, sums indexed by $|\tau| \in J$ for J an interval only refer to unplanted trees of that order. Planted trees will be explicitly denoted $\mathcal{I}(\tau)$.

We recall the remainder equation:

$$(\partial_t - \Delta)v = -v^3 - 3\sum_{w\in\mathcal{W}} \Upsilon(w) v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w)$$
$$-3\sum_{w_1,w_2\in\mathcal{W}} \Upsilon(w_1)\Upsilon(w_2) v \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w_2) + \sum_{\tau\in\partial\mathcal{W}} \Upsilon(\tau)\mathbb{X}_\bullet\tau.$$

Here we have introduced $\partial\mathcal{W} = \{\tau \in \mathcal{N}, \tau = \mathcal{I}(w_1)\mathcal{I}(w_2)\mathcal{I}(w_3), w_i \in \mathcal{W}\}$. Note that the product v^3 does not need to be expressed using the renormalised product $\circ_{\mathbb{X}}$ since v is of positive regularity. All the factors Υ in there are just combinatorial factors ± 1 which is why we omitted the subscript variable y.

The first thing we do is to convolve this equation with the kernel Ψ_L, and we obtain:

$$(\partial_t - \Delta)v_L = -v_L^3 + (v_L^3 - (v^3)_L) - 3\sum_{w\in\mathcal{W}} \Upsilon(w)(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w))_L \qquad (4.9.4)$$
$$-3\sum_{w_1,w_2\in\mathcal{W}} \Upsilon(w_1)\Upsilon(w_2)(v \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w_2))_L - \sum_{\tau\in\partial\mathcal{W}} \Upsilon(\tau)(\mathbb{X}_\bullet\tau)_L.$$

To apply the maximum principle 2.4.2, we will therefore need bounds on the commutator, which is easy enough:

$$|(v^3)_L - (v_L)^3| \leqslant \|v\|^2 [v]_\alpha L^\alpha,$$

as well as quantities of the type:

$$(\mathbb{X}\tau)_L(x) \text{ where } \tau \in \partial\mathcal{W},$$

which is bounded by $[\tau]_{|\tau|}L^{|\tau|} \leqslant c\|v\|_{D_d}^{\delta m_\Xi(\tau)} L^{-3+\delta m_\Xi(\tau)}$, in view of Assumption 4.9.6 and (4.3.1). The following two products will require more work:

$$(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w))_L \text{ with } w \in \mathcal{W}$$

and

$$(v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_2))_L \text{ with } w_1, w_2 \in \mathcal{W}$$

Since the path $\mathbb{X}$ is built such that two symmetric trees have the same image, studying those cases will be enough to be exhaustive.

We formulate the following assumption, for some $1 > c > 0$ to be fixed later in (4.10.16) and (4.10.22).

Assumption 4.9.6. *For all $\tau \in \mathcal{T}$ such that $m_\Xi(\tau) \neq 0$,*

$$[\mathbb{X}; \tau] \leqslant c\|v\|_D^{\delta m_\Xi(\tau)}. \tag{4.9.5}$$

Remark 4.9.7. Note that it is not necessary to do the proof of the main theorem by contradiction with this Assumption, but it simplifies greatly the computations and allows us to write everything in powers of $\|v\|$. Alternatively a proof can be made by keeping all norms of trees in the computation, but that becomes very messy fast.

Under Assumption 4.9.6, we have by Lemma 4.5.13 the following lemma:

Lemma 4.9.8. *For all $\tau \in \mathcal{T}$ such that $m_\Xi(\tau) \neq 0$,*

$$[\mathbb{X}; \mathcal{I}(\tau)] \lesssim c\|v\|_D^{\delta m_\Xi(\tau)}, \tag{4.9.6}$$

Proof. For $\tau \in \mathcal{W}$, it is immediate.
For $\tau \in \mathring{\mathcal{N}}$, one simply has to notice that for $\overline{\tau} \leqslant \tau$, $m_\Xi(\tau) = m_\Xi(\overline{\tau}) + m_\Xi(C_+(\overline{\tau}, \tau))$ and then use induction on $m_\Xi(\tau)$ to bound $[\mathbb{X}; C_+(\overline{\tau}, \tau)]$. □

With this set-up, we manage to prove the following lemmas, which hold for any domain D, uniformly over $x \in D$.

Lemma 4.9.9. *Under the Assumption 4.9.6, for $w_1, w_2 \in \mathcal{W}$, there exists an $\epsilon > 0$ such that for $J = [-2, -6 - |w_1| - |w_2| + \epsilon)$,*

$$\begin{aligned}\Big|(v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_2))_L(x) - \sum_{|\tau| \in J} \Big(\Upsilon_x(\tau) \mathbb{X}_{\bullet,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\Big)_L(x)\Big| \\ \lesssim c \sum_{|\tau| \in J} \|v\|_D^{\delta m_\Xi(\tau + w_1 + w_2)} [U^\tau]_{-6-|w_1|-|w_2|-|\tau|+\epsilon} L^\epsilon.\end{aligned} \tag{4.9.7}$$

We also have, for $|\tau| \in J$,

$$\begin{aligned}\Big|\Upsilon_x(\tau)\Big(\mathbb{X}_{\bullet,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\Big)_L(x)\Big| \\ \leqslant c\|v\|_D^{m_{\mathbf{1}}(\tau) + \delta m_\Xi(\tau + w_1 + w_2)} \|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\tau)} L^{6+|\tau|+|w_1|+|w_2|}.\end{aligned} \tag{4.9.8}$$

In this lemma we extend the notation m_Ξ to sums of trees linearly. The functions $m_{\mathbf{1}}, m_{\mathbf{x}}$ and m will also be extended similarly later.

Lemma 4.9.10. *Under the Assumption 4.9.6, for $w \in \mathcal{W}$, there exists $\epsilon > 0$ such that for $\widetilde{J} = \{(a,b) \in [-2,-1]^2, a+b < -6-|w|+\epsilon\}$,*

$$\begin{aligned}\Big|&(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w))_L(x) \\ &- \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\Big(\mathbb{X}_{\bullet,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\Big)_L(x)\Big| \\ &\lesssim c \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \|v\|_D^{\delta m_\Xi(\tau_1+\tau_2+w)} [U^{\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)}]_{-6-|w|-|\tau_1|-|\tau_2|+\epsilon} L^\epsilon.\end{aligned} \tag{4.9.9}$$

We also have, for $(|\tau_1|,|\tau_2|) \in \widetilde{J}$

$$\begin{aligned}\Big|\Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\Big(\mathbb{X}_{\bullet,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\Big)_L(x)\Big| \\ \leqslant c\|v\|_D^{m_{\mathbf{1}}(\tau_1+\tau_2)+\delta m_\Xi(\tau_1+\tau_2+w)} L^{6+|\tau_1|+|\tau_2|+|w|}.\end{aligned} \tag{4.9.10}$$

In both of these lemmas, the existence of the ϵ follows from the following remark.

Remark 4.9.11. Our choice of δ is such that $\mathcal{I}(\mathbf{1})\mathcal{I}(\mathbf{1})\mathcal{I}(\mathbf{1})$ is the only tree of order 0. Therefore for any non-trivial product, the sum can be indexed over trees τ of order $|\tau| < \epsilon$, for some $\epsilon > 0$. The renormalised product is therefore described up to positive order ϵ.

Applying the Schauder Lemma yields the following lemma:

Lemma 4.9.12. *Under the Assumption 4.9.6, for any $1 > d_0 > 0$ we have*

$$\sup_{d\leqslant d_0} d\|v_{\mathbf{X}}\|_{D_d} \lesssim \|v\|_D, \tag{4.9.11}$$

and

$$\sup_{d\leqslant d_0} d^{\gamma-\beta-|\tau|+m_{\mathbf{x}}(\tau)} [U^\tau]_{\gamma-\beta-|\tau|,D_d,d} \lesssim \|v\|_D^{m_{\mathbf{1}}(\tau)+m_{\mathbf{x}}(\tau)}. \tag{4.9.12}$$

A few more computations allow to close this argument, with a specification of d_0 in (4.10.13).

Lemma 4.9.13. *Under the Assumption 4.9.6, there exist $\lambda > 0$ such that*

$$\|v\|_{D_{\lambda\|v\|_D^{-1}}} \leqslant \frac{\|v\|_D}{2}. \tag{4.9.13}$$

The final proof of the main theorem relies on a iteration of this result. We define a finite

sequence $0 = R_0 < ... < R_N = 1$ by setting

$$R_{n+1} - R_n = \lambda \|v\|_{D_{R_n}}^{-1},$$

as long as the R_{n+1} defined that way stay less than 1. We terminate this algorithm once it would produce $R_{n+1} \geqslant 1$ in which case we set $R_N = R_{n+1} = 1$, or once Assumption 4.9.6 does not hold for $D' = D_{R_n}$. Note that $\|v\|_{D_{R_n}}^{-1}$ is strictly increasing so the sequence necessarily terminates after finitely many steps. Rewriting Lemma 4.9.13 replacing D by D_{R_n} then gives the bounds for smaller and smaller boxes.

$$\|v\|_{D_{R_{n+1}}} \leqslant \frac{\|v\|_{D_{R_n}}}{2}$$

We now prove that Theorem 4.9.1 holds for all $d = R_n, n = 0...N$. If Assumption 4.9.6 does not hold for N then it is immediate, else for $k < n$, $\|v\|_{D_{R_n}} \leqslant \|v\|_{D_{R_k}} 2^{k-n}$ hence

$$R_n = \sum_{k=0}^{n-1} R_{k+1} - R_k = \sum_{k_0}^{n-1} \lambda \|v\|_{D_{R_k}}^{-1} \leqslant \lambda \|v\|_{D_{R_{n-1}}}^{-1} \sum_{k_0}^{n-1} 2^{k-n+1} \lesssim \|v\|_{D_{R_{n-1}}}^{-1}.$$

This implies that for any $R \in (R_{n-1}, R_n)$, $\|v\|_{D_R} \leqslant \|v\|_{D_{R_{n-1}}} \lesssim R_n^{-1} \leqslant R^{-1}$, which proves the theorem in that case.

If the end-point is $R_N = 1$, we either have $R_{N-1} > \frac{1}{2}$ or $R_N - R_{N-1} > \frac{1}{2}$. In both cases $\|v\|_{D_{R_{N-1}}} \lesssim R_{N-1}^{-1} \lesssim 1$.

4.10 Proof of the intermediate results

4.10.1 A technical lemma

We first quantify the expansions given in equations (4.6.8),(4.6.9) and (4.6.10), used now with $\Theta = \Upsilon$.

$$[V]_\alpha = \sup_{x,y} \frac{|v_{\mathbf{1}}(x) - V_\alpha(y,x)|}{d(x,y)^\alpha},$$
$$[V^2]_\alpha = \sup_{x,y} \frac{|v_{\mathbf{1}}(x)^2 - V_\alpha^2(y,x)|}{d(x,y)^\alpha},$$
$$[V^{(i)}]_\alpha = \sup_{x,y} \frac{|v_{\mathbf{X}_i}(x) - V_\alpha^{(i)}(y,x)|}{d(x,y)^\alpha}.$$

For any domain D, we denote the restriction of this norm to $x, y \in D$ by adding the subscript D. A second subscript d may be added when we restrict to x, y satisfying

$d(x,y) < d$.

Using Theorem 4.6.10 we have the identities.

$$[U^\tau]_{\gamma-|\tau|} = \begin{cases} [V]_{\gamma-|\tau|} & \text{if } m_{\mathbf{1}}(\tau) = 1, \\ [V^2]_{\gamma-|\tau|} & \text{if } m_{\mathbf{1}}(\tau) = 2, \\ [V^{(i)}]_{\gamma-|\tau|} & \text{if } m_{\mathbf{x}_i}(\tau) = 1, \\ 0 & \text{if } m_{\mathbf{1}}(\tau), m_{\mathbf{x}}(\tau) = 0\ . \end{cases} \tag{4.10.1}$$

Using Lemma 4.6.5 and the Assumption 4.9.6 to replace all order bounds on trees in this lemma by powers of $\|v\|_D$, we get the following general bound for the norm of U. The bound in the case of $m_{\mathbf{x}}(\tau) = 1$ is a straightforward application of Lemma 3.2.5.

Lemma 4.10.1. *Under the Assumption 4.9.6, for any $\tau \in \mathcal{N}$, and $0 < \beta < \gamma < 2$,*

$$\begin{aligned} &\sup_{d \leqslant d_0} d^{\gamma-\beta+2-\delta m_\Xi(\tau)} [U^\tau]_{\gamma-\beta-|\tau|, D_d, d} \\ &\quad \lesssim \sup_{d \leqslant d_0} \Big[[V]_{\gamma, D_d, d} d^\gamma + \mathbb{1}_{\{\gamma-\beta+2-\delta m_\Xi(\tau) < 1\}} d \|v_{\mathbf{X}}\|_{D_d} \\ &\quad + c \sum_{\gamma-\beta-\delta m_\Xi(\tau) \leqslant |\overline{\tau}| < \gamma-2} d^{|\overline{\tau}|+2} \|v\|_D^{m_{\mathbf{1}}(\overline{\tau}) + m_\Xi(\overline{\tau})\delta} \|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\overline{\tau})} \Big]. \end{aligned} \tag{4.10.2}$$

4.10.2 Proof of Lemma 4.9.9

Take $w_1, w_2 \in \mathcal{W}$. From Definition 4.7.1, there exists $\epsilon > 0$ such that for $J = [-2, -6 - |w_1| - |w_2| + \epsilon)$,

$$\begin{aligned} &(v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_2))_L(x) = \\ &\qquad \sum_{|\tau| \in J} \Big(\Upsilon_\bullet(\tau) \mathbb{X}_{\bullet,\bullet}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2)) \Big)_L(x). \end{aligned} \tag{4.10.3}$$

We know J is the right interval even though we have a longer expansion of v because the unplanted trees of positive homogeneity vanish in our formalism. This corresponds to $|\tau| + |w_1| + |w_2| + 6 < 0$, and Remark 4.9.11 tells us that this expansion is the same to positive level ϵ, for $\epsilon > 0$ small enough.

We prove estimate (4.9.7) by using the reconstruction Lemma 3.2.8. Define $F(y,x) = \sum_{|\tau| \in J} \Upsilon_x(\tau) \mathbb{X}_{y,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))$, and we aim to bound a suitable regularisation of $F(y,x) - F(x,x)$. Lemma 3.2.8 and Assumption 4.9.6 imply the desired estimate

(4.9.7) as soon as the following identity is established:

$$\Big|\int \Psi_l(x_2-y)(F(y,x_1)-F(y,x_2))dy\Big| \tag{4.10.4}$$
$$\leqslant \sum_{|\tau|\in J}[\mathbb{X};\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2)][U^\tau]_{-6-|w_1|-|w_2|-|\tau|+\epsilon}$$
$$\times\, l^{6+|w_1|+|w_2|+|\tau|}d(x_1,x_2)^{-6-|w_1|-|w_2|-|\tau|+\epsilon}\,.$$

By multiplicativity of the coproduct, and since $w_1, w_2 \in \mathcal{W}$, we first note that for $\overline{\tau} \in \mathcal{N}$

$$\begin{aligned}\Delta(\mathcal{I}(\overline{\tau})\mathcal{I}(w_1)\mathcal{I}(w_2)) &= \Delta\mathcal{I}(\overline{\tau})\Delta\mathcal{I}(w_1)\Delta\mathcal{I}(w_2)\\ &= \sum_{-2\leqslant|\tau|\leqslant|\overline{\tau}|}\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2)\otimes C_+(\tau,\overline{\tau}).\end{aligned}$$

Using Chen's relation, we have:

$$\begin{aligned}F(y,x_1) &= \sum_{|\overline{\tau}|\in J}\Upsilon_{x_1}(\overline{\tau})\mathbb{X}_{y,x_1}(\mathcal{I}(\overline{\tau})\mathcal{I}(w_1)\mathcal{I}(w_2))\\ &= \sum_{|\overline{\tau}|\in J}\Upsilon_{x_1}(\overline{\tau})\sum_{|\tau|\in J}\mathbb{X}_{y,x_2}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\mathbb{X}_{x_2,x_1}C_+(\tau,\overline{\tau})\\ &= \sum_{|\tau|\in J}\mathbb{X}_{y,x_2}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\sum_{|\overline{\tau}|\in J}\Upsilon_{x_1}(\overline{\tau})\mathbb{X}_{x_2,x_1}C_+(\tau,\overline{\tau}).\end{aligned}$$

Therefore,

$$\begin{aligned}&F(y,x_1)-F(y,x_2)\\ &= \sum_{|\tau|\in J}\mathbb{X}_{y,x_2}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\Big(\sum_{|\overline{\tau}|\in J}\Upsilon_{x_1}(\overline{\tau})\mathbb{X}_{x_2,x_1}C_+(\tau,\overline{\tau})-\Upsilon_{x_2}(\tau)\Big)\\ &= -\sum_{|\tau|\in J}\mathbb{X}_{y,x_2}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))U^\tau_{-6-|w_1|-|w_2|-|\tau|+\epsilon}(x_2,x_1)\end{aligned}$$

which proves (4.10.4) and thus (4.9.7).

The bound (4.9.8) is simply the order bound on the trees, which for $x \in D_d$ and for $L < d$, can be expressed as:

$$|\Upsilon_x(\tau)\Big(\mathbb{X}_{\bullet,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\Big)_L(x)| \tag{4.10.5}$$
$$\leqslant \|v\|_D^{m_{\mathbf{1}}(\tau)}\|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\tau)}[\mathbb{X};\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2)]L^{6+|\tau|+|w_1|+|w_2|}$$
$$\overset{\text{Ass. 4.9.6}}{\leqslant} c\|v\|_D^{m_{\mathbf{1}}(\tau)+\delta m_\Xi(\tau+w_1+w_2)}\|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\tau)}L^{6+|\tau|+|w_1|+|w_2|}.$$

4.10.3 Proof of Lemma 4.9.10

Take $w \in \mathcal{W}$. From Definition 4.7.1 and Remark 4.9.11, there exists $\epsilon > 0$ such that for $\widetilde{J} = \{(a,b) \in [-2,-1]^2, a+b < -6-|w|+\epsilon\}$,

$$\begin{aligned}&(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w))_L(x)\\ &= \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \Big(\Upsilon(\tau_1)\Upsilon(\tau_2)\mathbb{X}_{\bullet,\bullet}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\Big)_L(x). \end{aligned} \tag{4.10.6}$$

We know J is the right domain even though we have a longer expansion for v because unplanted trees of positive order vanish in our setting.

We prove the bound (4.9.9).

Define $F(y,x) = \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\mathbb{X}_{y,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))$, and we aim to bound a suitable regularisation of $F(y,x)-F(x,x)$. Lemma 3.2.8 and Assumption 4.9.6 implies the desired bound as soon as the following bound is established:

$$\begin{aligned}&\Big|\int \Psi_l(x_2-y)(F(y,x_1)-F(y,x_2))dy\Big| \leqslant\\ &\qquad \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} [\mathbb{X};\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)][U^{\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)}]_{-6-|w|-|\tau_1|-|\tau_2|+\epsilon}\\ &\qquad \times l^{6+|\tau_1|+|\tau_2|+|w|} d(x_1,x_2)^{-6-|w|-|\tau_1|-|\tau_2|+\epsilon}\end{aligned} \tag{4.10.7}$$

By multiplicativity of the coproduct, and since $w \in \mathcal{W}$, we first note that for $(|\overline{\tau}_1|,|\overline{\tau}_2|) \in \widetilde{J}$,

$$\begin{aligned}\Delta(\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(w)) &= \Delta\mathcal{I}(\overline{\tau}_1)\Delta\mathcal{I}(\overline{\tau}_2)\Delta\mathcal{I}(w)\\ &= \sum_{\substack{-2\leqslant|\tau_2|\leqslant|\overline{\tau}_2|\\ -2\leqslant|\tau_1|\leqslant|\overline{\tau}_1|}} \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)\otimes C_+(\tau_1,\overline{\tau}_1)C_+(\tau_2,\overline{\tau}_2).\end{aligned}$$

Using Chen's relation, we have

$$\begin{aligned}&F(y,x_1)\\ &= \sum_{(|\overline{\tau}_1|,|\overline{\tau}_2|)\in\widetilde{J}} \Upsilon_{x_1}(\overline{\tau}_1)\Upsilon_{x_1}(\overline{\tau}_2)\mathbb{X}_{y,x_1}(\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(w))\\ &= \sum_{(|\overline{\tau}_1|,|\overline{\tau}_2|)\in\widetilde{J}} \Upsilon_{x_1}(\overline{\tau_1})\Upsilon_{x_1}(\overline{\tau_2})\\ &\qquad \times \sum_{\substack{-2\leqslant|\tau_2|\leqslant|\overline{\tau}_2|\\ -2\leqslant|\tau_1|\leqslant|\overline{\tau}_1|}} \mathbb{X}_{y,x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\mathbb{X}_{x_2,x_1}(C_+(\tau_1,\overline{\tau}_1)C_+(\tau_2,\overline{\tau}_2))\end{aligned}$$

$$= \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \mathbb{X}_{y,x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)) \times \sum_{(|\overline{\tau}_1|,|\overline{\tau}_2|)\in\widetilde{J}} \Upsilon_{x_1}(\overline{\tau}_1)\Upsilon_{x_1}(\overline{\tau}_2)\mathbb{X}_{x_2,x_1}(C_+(\tau_1,\overline{\tau}_1)C_+(\tau_2,\overline{\tau}_2)).$$

In the following computation, we introduce a mock $\Upsilon(\Xi)$, which is just a factor -1, and $C_+(\Xi,\Xi) = 1$ to make explicit that the structure of the terms appearing here is that of $U^{\widetilde{\tau}}_{\beta}$ for some $\widetilde{\tau}$ and β.

$$\begin{aligned}
F(y,x_1) - F(y,x_2) &= \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \mathbb{X}_{y,x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)) \\
&\times \Big(\sum_{(|\overline{\tau}_1|,|\overline{\tau}_2|)\in\widetilde{J}} \Upsilon_{x_1}(\overline{\tau}_1)\Upsilon_{x_1}(\overline{\tau}_2)\mathbb{X}_{x_2,x_1}(C_+(\tau_1,\overline{\tau}_1)C_+(\tau_2,\overline{\tau}_2)) - \Upsilon_{x_2}(\tau_1)\Upsilon_{x_2}(\tau_2)\Big) \\
&= \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \mathbb{X}_{y,x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)) \times \Big(\Upsilon_{x_2}(\tau_1)\Upsilon_{x_2}(\tau_2)\Upsilon_{x_2}(\Xi) \\
&- \sum_{(|\overline{\tau}_1|,|\overline{\tau}_2|)\in\widetilde{J}} \Upsilon_{x_1}(\overline{\tau}_1)\Upsilon_{x_1}(\overline{\tau}_2)\Upsilon_{x_1}(\Xi)\mathbb{X}_{x_2,x_1}(C_+(\tau_1,\overline{\tau}_1)C_+(\tau_2,\overline{\tau}_2)C_+(\Xi,\Xi))\Big) \\
&= \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \mathbb{X}_{y,x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)) \times \Big(\Upsilon_{x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)) \\
&- \sum_{(|\overline{\tau}_1|,|\overline{\tau}_2|)\in\widetilde{J}} \Upsilon_{x_1}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi))\mathbb{X}_{x_2,x_1}(C_+(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi),\mathcal{I}(\overline{\tau}_1)\mathcal{I}(\overline{\tau}_2)\mathcal{I}(\Xi))\Big) \\
&= \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \mathbb{X}_{y,x_2}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))U^{\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)}_{-3+\delta-|w|+\epsilon},
\end{aligned}$$

which proves (4.10.7) and thus (4.9.9).

The bound (4.9.10) is directly the order bound on the trees, which for $x \in D_d$ and for $L < d$, can be expressed as:

$$\begin{aligned}
|\Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\Big(\mathbb{X}_{\bullet,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\Big)_L(x)| \qquad (4.10.8)\\
&\leqslant \|v\|_D^{m_{\mathbf{1}}(\tau_1+\tau_2)}[\mathbb{X};\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)]L^{6+|\tau_1|+|\tau_2|+|w|} \\
&\overset{\text{Ass. 4.9.6}}{\leqslant} c\|v\|_D^{m_{\mathbf{1}}(\tau_1+\tau_2)+\delta m_\Xi(\tau_1+\tau_2+w)}L^{6+|\tau_1|+|\tau_2|+|w|}.
\end{aligned}$$

Note that here the term $\|v_{\mathbf{X}}\|$ does not appear since $\widetilde{J}$ does not contain any homogeneities higher than 1.

4.10.4 Proof of Lemma 4.9.12

For $\gamma \in (2-2\delta, 2)$ we have

$$(\partial_t - \Delta)_y V_\gamma(y,x) = \sum_{-2<|\tau|<\gamma-2} \Upsilon_x(\tau)\mathbb{X}_{yx}(\tau).$$

We write trees in this sum as $\tau = \mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\tau_3)$. The first remark we make is that if $|\tau_i| \geqslant -2$ for $i = 1,2,3$, then $|\tau| \geqslant 0 > \gamma - 2$. We also remark that for $w \in \mathcal{W}$, $\Upsilon_x(w)$ is independent of x and for $\tau \in \partial\mathcal{W}$, $\mathbb{X}_{yx}\tau$ is also independent of x. Therefore, accounting for symmetries with the factor 3, we get:

$$\begin{aligned}
&(\partial_t - \Delta)_y V_\gamma(y,x) = \\
&\quad -3\sum_{w\in\mathcal{W}} \Upsilon_y(w) \sum_{|\tau_1|+|\tau_2|<\gamma-8-|w|} \Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\mathbb{X}_{y,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w)) \\
&\quad -3\sum_{w_1,w_2\in\mathcal{W}} \Upsilon_y(w_1)\Upsilon_y(w_2) \sum_{|\tau|<\gamma-8-|w_1|-|w_2|} \Upsilon_x(\tau)\mathbb{X}_{y,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2)) \\
&\quad + \sum_{\tau\in\partial\mathcal{W}} \Upsilon_y(\tau)\mathbb{X}_{y,y}\tau.
\end{aligned}$$

Using the remainder equation, we have:

$$\begin{aligned}
&(\partial_t - \Delta)(v - V_\gamma(\cdot,x))(y) = -v^3(y) \\
&\quad -3\sum_{w\in\mathcal{W}} \Upsilon_y(w)\Big((v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_y\mathcal{I}(w))(y) \\
&\qquad\qquad - \sum_{|\tau_1|+|\tau_2|<\gamma-8-|w|} \Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\mathbb{X}_{y,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\Big) \\
&\quad -3\sum_{w_1,w_2\in\mathcal{W}} \Upsilon_y(w_1)\Upsilon_y(w_2)\Big((v \circ_{\mathbb{X}} \mathbb{X}_y\mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_y\mathcal{I}(w_2))(y) \\
&\qquad\qquad - \sum_{|\tau|<\gamma-8-|w_1|-|w_2|} \Upsilon_x(\tau)\mathbb{X}_{y,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\Big).
\end{aligned}$$

We need to bound this after integration against $\Psi_{L_1}(z-y)dy$ for $z \in B(x, L_2)$, for $x \in D_{2d}$, for $L_1 < \frac{d}{2}$ and $L_2 < \frac{d}{4}$ to apply the Schauder Lemma 3.2.3. We first have:

$$|(v^3)_{L_1}(z)| \leqslant \|v\|_D^3.$$

If $F(y,x) = \sum_{|\tau_1|+|\tau_2|<\gamma-8-|w|} \Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\mathbb{X}_{y,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))$, then Lemma 4.9.10 gives a bound on $(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w) - F(\bullet, z))_{L_1}(z)$, and from equation (4.10.7) we

have a bound on $(F(\bullet,x)-F(\bullet,z))_{L_1}(z)$. Together with Assumption 4.9.6, they give

$$\begin{aligned}&\Big|\Big((v\circ_{\mathbb{X}} v\circ_{\mathbb{X}} \mathbb{X}_\bullet\mathcal{I}(w)) \\ &-\sum_{|\tau_1|+|\tau_2|<\gamma-8-|w|}\Upsilon_x(\tau_1)\Upsilon_x(\tau_2)\mathbb{X}_{\cdot,x}(\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(w))\Big)_{L_1}(z)\Big| \\ &\lesssim c\sum_{|\tau_1|+|\tau_2|<\gamma-8-|w|}[U^{\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)}]_{-6-|w|-|\tau_1|-|\tau_2|+\epsilon,D_d,d} \\ &\quad\times\|v\|_D^{\delta m_\Xi(\tau_1+\tau_2+w)}(L_1^\epsilon+L_1^{6+|\tau_1|+|\tau_2|+|w|}d(x,z)^{-6-|w|-|\tau_1|-|\tau_2|+\epsilon}).\end{aligned} \tag{4.10.9}$$

Similarly with $F(y,x)=\sum_{|\tau|<\gamma-8-|w_1|-|w_2|}\Upsilon_x(\tau)\mathbb{X}_{y,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))$, Lemma 4.9.9 gives a bound on $(v\circ_{\mathbb{X}}\mathbb{X}_\bullet\mathcal{I}(w_1)\circ_{\mathbb{X}}\mathbb{X}_\bullet\mathcal{I}(w_2))-F(\bullet,z))_{L_1}(z))$ and from Equation (4.10.4), we have a bound on $(F(\bullet,x)-F(\bullet,z))_{L_1}(z)$. Together with Assumption 4.9.6, they give

$$\begin{aligned}&\Big|\Big(v\circ_{\mathbb{X}}\mathbb{X}_\bullet\mathcal{I}(w_1)\circ_{\mathbb{X}}\mathbb{X}_\bullet\mathcal{I}(w_2) \\ &\qquad-\sum_{|\tau|<\gamma-8-|w_1|-|w_2|}\Upsilon_x(\tau)\mathbb{X}_{\bullet,x}(\mathcal{I}(\tau)\mathcal{I}(w_1)\mathcal{I}(w_2))\Big)_L(z)\Big| \\ &\lesssim c\sum_{|\tau|<\gamma-8-|w_1|-|w_2|}[U^\tau]_{-6-|w_1|-|w_2|-|\tau|+\epsilon,D_d,d} \\ &\quad\times\|v\|_D^{\delta m_\Xi(\tau+w_1+w_2)}(L^\epsilon+L^{6+|w_1|+|w_2|+|\tau|}d(x,z)^{-6-|w_1|-|w_2|-|\tau|+\epsilon}).\end{aligned} \tag{4.10.10}$$

We also need the three-point continuity. It is a consequence of Lemma 4.6.4, and can be quantified, for $x\in D_d$, for $y\in B(x,\frac{d}{4})$, for $z\in B(y,\frac{d}{4})$, as:

$$\begin{aligned}&|V_\gamma(z,x)-V_\gamma(z,y)+V_\gamma(y,y)-V_\gamma(y,x)+V_\gamma^i(y,x)(z_i-y_i)| \\ &\qquad\leqslant\sum_{\substack{-2<|\tau|<\gamma-2\\ \tau\neq\mathbf{X}}}[U^\tau]_{\gamma-2-|\tau|,D_{\frac{d}{2}},\frac{d}{2}}d(y,x)^{\gamma-2-|\tau|}\,[\mathbb{X};\mathcal{I}(\tau)]d(z,y)^{|\tau|+2} \\ &\overset{\text{Ass. 4.9.6}}{\leqslant} c\sum_{\substack{-2<|\tau|<\gamma-2\\ \tau\neq\mathbf{X}}}\|v\|_D^{\delta m_\Xi(\tau)}[U^\tau]_{\gamma-2-|\tau|,D_{\frac{d}{2}},\frac{d}{2}}d(y,x)^{\gamma-2-|\tau|}\,d(z,y)^{|\tau|+2}.\end{aligned} \tag{4.10.11}$$

We now notice that all the term appearing in (4.10.9),(4.10.10) and (4.10.11) have a common structure. By replacing the homogeneities by their expressions in terms of δm_Ξ for the trees in $\mathcal{W}$, and relabelling $\tau=\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)$ in (4.10.9), we get after application of Lemma 3.2.3:

$$\begin{aligned}\sup_{d\leqslant d_0}d^\gamma[V]_{\gamma,D_d}&\lesssim\sup_{d\leqslant d_0}d^2\|v\|_D^3 \\ &\quad+c\sup_{a,b,\tau}\sup_{d\leqslant d_0}\Big(d^b\|v\|_{D_d}^{a+\delta m_\Xi(\tau)}[U^\tau]_{b-2-a-|\tau|,D_d,d}\Big),\end{aligned} \tag{4.10.12}$$

where $[V]_{\gamma,D_d}$ is the γ-Hölder norm of V_γ, restricted to the domain D_d, and where the supremum is taken over a finite subset of

$$\{(a,b,\tau) \in \mathbb{R}^2_+ \times \mathcal{T}_r, a \geqslant 0\ , b \geqslant \gamma\ , b-2-a-|\tau| < \gamma\}.$$

We apply Lemma 4.10.1 to the second part.

$$\begin{aligned}\sup_{d\leqslant d_0} d^\gamma [V]_{\gamma,D_d} &\lesssim \sup_{d\leqslant d_0} d^2 \|v\|_D^3 \\ &+ c \sup_{a,b,\tau} \sup_{d\leqslant d_0} d^{a+\delta m_\Xi(\tau)} \|v\|_{D_d}^{a+\delta m_\Xi(\tau)} \Big([V]_{\gamma,D_d} d^\gamma + \mathbb{1}_{\{b-a-\delta m_\Xi(\tau)<1\}} d \|v_{\mathbf{X}}\|_{D_d} \\ &\qquad + c \sum_{b-a-2-\delta m_\Xi(\tau)\leqslant|\overline{\tau}|<\gamma-2} d^{|\overline{\tau}|+2} \|v\|_D^{m_{\mathbf{1}}(\overline{\tau})+\delta m_\Xi(\overline{\tau})} \|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\overline{\tau})} \Big).\end{aligned}$$

We see now that if we take

$$d_0 = \|v\|_D^{-1} \tag{4.10.13}$$

, then there exist a value of $c_0 < 1$ such that for any $0 < c < c_0$, the occurrences of $[V]_{\gamma,D_d}$ can be absorbed into the left-hand side, and the other terms also simplify: in the last sum, if $m_{\mathbf{x}}(\overline{\tau}) = 1$, we bound

$$d^{|\overline{\tau}|+2} \|v\|_D^{m_{\mathbf{1}}(\overline{\tau})+\delta m_\Xi(\overline{\tau})} \|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\overline{\tau})} \leqslant d^{-1+2m_{\mathbf{x}}(\overline{\tau})} \|v_{\mathbf{X}}\|_{D_d}^{m_{\mathbf{x}}(\overline{\tau})} = d\|v_{\mathbf{X}}\|_{D_d},$$

and if $m_{\mathbf{x}}(\overline{\tau}) = 0$,

$$d^{|\overline{\tau}|+2} \|v\|_D^{m_{\mathbf{1}}(\overline{\tau})+\delta m_\Xi(\overline{\tau})} = d^{-1+m_{\mathbf{1}}(\overline{\tau})+\delta m_\Xi(\overline{\tau})} \|v\|_D^{m_{\mathbf{1}}(\overline{\tau})+\delta m_\Xi(\overline{\tau})} \leqslant \|v\|_D.$$

In conclusion,

$$\sup_{d\leqslant d_0} d^\gamma [V]_{\gamma,D_d} \lesssim \|v\|_D + c \sup_{d\leqslant d_0} d\|v_{\mathbf{X}}\|_{D_d}. \tag{4.10.14}$$

We now prove the bound on $\|v_{\mathbf{X}}\|$. For that we take ϵ small enough such that there is no tree of regularity between 1 and $1+\epsilon$. Then we can apply Corollary 3.2.5 with $\kappa = 1+\epsilon$ but with $U(x,y) = \sum_{-2\leqslant|\tau|<-1} \Upsilon_x(\tau)\mathbb{X}_{yx}\mathcal{I}(\tau)$. We get

$$\|v_{\mathbf{X}}\|_{D_d} \lesssim [V]_{1+\epsilon,D_d,d} d^\epsilon + \|U\|_{D_d,d} d^{-1}.$$

We have by Assumption 4.9.6

$$\|U\|_{D_d,d} d^{-1} \lesssim \sum_n d^{n-2} \|v\|_D^n,$$

and from (4.10.2)

$$[V]_{1+\epsilon,D_d,d}d^{\epsilon} \lesssim [V]_{\gamma,D_d,d}d^{\gamma-1} + c\sum_{n,m} d^{n+2m-2}\|v\|_{D_d}^n\|v_{\mathbf{X}}\|_{D_d}^m,$$

where the sum ranges over a finite set of indices $n \geqslant 0$ and $m \in \{0,1\}$. We have, assuming $d_0 \leqslant \|v\|_D^{-1}$,

$$\sup_{d\leqslant d_0} d\|v_{\mathbf{X}}\|_{D_d} \lesssim \sup_{d\leqslant d_0}\Big([V]_{\gamma,D_d,d}d^{\gamma} + c(\|v\|_D + d\|v_{\mathbf{X}}\|_{D_d})\Big).$$

If we take c small enough, depending on the constant implicit in $\lesssim$, for some constant $C > 0$ we have,

$$\sup_{d\leqslant d_0} d\|v_{\mathbf{X}}\|_{D_d} \leqslant C\sup_{d\leqslant d_0}\Big([V]_{\gamma,D_d,d}d^{\gamma} + \|v\|_D\Big). \tag{4.10.15}$$

Together with (4.10.14), this gives, for a constant c small enough,

$$\sup_{d\leqslant d_0} d^{\gamma}[V]_{\gamma,D_d,d} \lesssim \|v\|_D. \tag{4.10.16}$$

4.10.5 Proof of Lemma 4.9.13

We apply the Lemma 2.4.2 to the convolved equation (4.9.4), on the domain D.

$$\begin{aligned}\|v\|_{D_{d+d'}} \lesssim \max\Big\{&d'^{-1}, \quad \|(v^3)_L - v_L^3\|_{D_d}^{\frac{1}{3}}, \quad \|(\mathbb{X}\tau)_L\|_{D_d}^{\frac{1}{3}}, \quad \tau \in \partial\mathcal{W},\\ &\|(v\circ_{\mathbb{X}}\mathbb{X}_{\cdot}\mathcal{I}(w_1)\circ_{\mathbb{X}}\mathbb{X}_{\cdot}\mathcal{I}(w_2))_L\|_{D_d}^{\frac{1}{3}},\ w_i\in\mathcal{W},\\ &\|(v\circ_{\mathbb{X}} v\circ_{\mathbb{X}}\mathbb{X}_{\bullet}\mathcal{I}(w))_L\|_{D_d}^{\frac{1}{3}},\ w\in\mathcal{W}, \quad \|v - v_L\|_{D_{d+d'}}\Big\}\end{aligned} \tag{4.10.17}$$

We have for $d > L$

$$\begin{aligned}\|(v^3)_L - v_L^3\|_{D_d} &\lesssim \|v\|_{D_{d-L}}^2[v]_{\alpha,D_{d-L},L}L^{\alpha} \qquad \text{and}\\ \|v - v_L\|_{D_{d+d'}} &\leqslant [v]_{\alpha,D_{d-L},L}L^{\alpha}\end{aligned} \tag{4.10.18}$$

and for α small enough, we have by Lemma 4.9.12, $[v]_{\alpha,D_{d-L},L} \lesssim (d-L)^{-\alpha}\|v\|_D$.

From Lemma 4.9.9 we get for $w_1, w_2 \in \mathcal{W}$,

$$\begin{aligned}&\|(v\circ_{\mathbb{X}}\mathbb{X}_{\bullet}\mathcal{I}(w_1)\circ_{\mathbb{X}}\mathbb{X}_{\bullet}\mathcal{I}(w_2))_L\|_{D_d}\\ &\quad\lesssim c\sum_{|\tau|\in J}\|v\|_D^{\delta m_{\Xi}(\tau+w_1+w_2)}\Big([U^{\tau}]_{-6-|w_1|-|w_2|-|\tau|+\epsilon,D_{d-L},d}L^{\epsilon}\\ &\quad+\|v\|_D^{m_{\mathbf{1}}(\tau)}\|v_{\mathbf{X}}\|_{D_{d-L}}^{m_{\mathbf{x}}(\tau)}L^{6+|\tau|+|w_1|+|w_2|}\Big).\end{aligned}$$

From Lemma 4.9.10, we get for $w \in \mathcal{W}$,

$$\begin{aligned}
&\|(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w))_L\|_{D_d} \\
&\lesssim c \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \|v\|^{\delta m_\Xi(\tau_1+\tau_2+w)} \Big([U^{\mathcal{I}(\tau_1)\mathcal{I}(\tau_2)\mathcal{I}(\Xi)}]_{-6-|w|-|\tau_1|-|\tau_2|+\epsilon, D_{d-L}, d} L^\epsilon \\
&\quad + \|v\|_D^{m_{\mathbf{1}}(\tau_1+\tau_2)} L^{6+|\tau_1|+|\tau_2|+|w|} \Big).
\end{aligned}$$

Using Lemma 4.9.12 and setting $d = \|v\|_D^{-1}$ and $L = \frac{d}{k}$ for some $k \geqslant 2$ gives for $w_1, w_2 \in \partial\mathcal{W}$,

$$\|(v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_1) \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w_2))_L\|_{D_d} \lesssim c\|v\|_D^3 K(w_1, w_2, k) \,, \tag{4.10.19}$$

where

$$K(w_1, w_2, k) = \sum_{|\tau|\in J} \Big(k^{-\epsilon} + k^{3-\delta m_\Xi(\tau+w_1+w_2)-m_{\mathbf{1}}(\tau)-2m_{\mathbf{x}}(\tau)} \Big),$$

and for $w \in \mathcal{W}$,

$$\|(v \circ_{\mathbb{X}} v \circ_{\mathbb{X}} \mathbb{X}_\bullet \mathcal{I}(w))_L\|_{D_d} \lesssim c\|v\|_D^3 K'(w, k) \tag{4.10.20}$$

where

$$K'(w, k) = \sum_{(|\tau_1|,|\tau_2|)\in\widetilde{J}} \Big(k^{-\epsilon} + k^{3-\delta m_\Xi(\tau_1+\tau_2+w)-m_{\mathbf{1}}(\tau_1+\tau_2)} \Big).$$

Finally, for $\tau \in \partial\mathcal{W}$, we get from Assumption 4.9.6

$$\|(\mathbb{X}\tau)_L\|_{D_d} \lesssim cL^{-3+\delta m_\Xi(\tau)} \|v\|_D^{\delta m_\Xi(\tau)} = c\|v\|_D^3 k^{3-\delta m_\Xi(\tau)}. \tag{4.10.21}$$

With (4.10.18), (4.10.19), (4.10.20) and (4.10.21), the bound (4.10.17) becomes

$$\begin{aligned}
\|v\|_{D_{\|v\|_D^{-1}+R}} \lesssim \max \Big\{ & R^{-1}, \quad k^{-\frac{\alpha}{3}} \|v\|_D, \quad c^{\frac{1}{3}} \|v\|_D k^{1-\frac{\delta m_\Xi(\tau)}{3}}, \ \tau \in \partial\mathcal{W}, \\
& c^{\frac{1}{3}} \|v\|_D K(w_1, w_2, k)^{\frac{1}{3}}, \ w_1, w_2 \in \mathcal{W}, \\
& c^{\frac{1}{3}} \|v\|_D K'(w, k)^{\frac{1}{3}}, \ w \in \mathcal{W}, \quad k^{-\alpha} \|v\|_D, \Big\}.
\end{aligned} \tag{4.10.22}$$

We see that we can now choose $k > 2$ large enough and then $c < c_0$, as well as $R = (\lambda - 1)\|v\|_D^{-1}$ for λ large enough such that (4.10.22) becomes:

$$\|v\|_{D_{\lambda\|v\|_D^{-1}}} \leqslant \frac{\|v\|_D}{2}. \tag{4.10.23}$$

Chapter 5

Conclusions and Future Work

5.1 Conclusion

In this thesis we have explored part of the potential of the deterministic technique developed in the first chapter. The particular maximum principle that comes with the strong non-linear term interacts perfectly with the convolution method to separate the analysis on large and small scales. In the first chapter, we introduced this new maximum principle and explored its limits in a setting where the noise was just rough enough to make the problem non-trivial. We were not able to translate the full range of the maximum principle to the irregular case unfortunately, and in particular the exponential case is not covered by this approach. We were however able to get some exponential integrability in the case of Gaussian noise.

The convolution method proved reliable enough to be applied in a simple case where renormalisation is needed in the second chapter. We decided to skip the traditional first step of the Φ^4 equation in dimension 2 and to go directly to dimension 3, since in this case we can still handle all the renormalisation terms by hand. In this section, we started to introduce our kernel-free version of the theory of regularity structures with in particular a Reconstruction lemma with an extremely simplified proof, and tailor-made Schauder theory with local expansions around a base-point. The probabilistic estimates in the first chapter gave a precious insight into what bound we could expect, and that simplified the proof. We were able to recover stretched exponential integrability, again in the case of Gaussian noise.

In the last chapter, we worked towards a systematic implementation of the convolution method. We defined a kernel-free approach to the theory of regularity structures in a more systematic way, although still not as general as Hairer's works. Our approach was designed to allow for the same localisation as in the previous chapters. In the algebraic

construction we proved a fundamental coherence result: using the coefficient map given by the fixed-point argument, continuity condition for different levels of the expansion are consequences of the bound given by the Schauder estimate. This result was not apparent nor necessary in dimension 3. In the analytic part of this chapter, we once again made use of the insight provided by probabilistic estimates, and in the case of Gaussian noise, stretched exponential integrability holds.

This thesis contains the first results on large scale behaviour of solutions to rough PDEs. This was made possible with a strong maximum principle, a localised Schauder theory and a restatement of the theory of regularity structures compatible with these localisation properties. We have however uncovered only a small amount of the potentials of the theory of regularity structures: the dynamic Φ^4 model is now considered to be one of the most simple examples of application of the theory, at least up to dimension 3.

5.2 Future Work

The obvious follow-up to this thesis is therefore the extension of this convolution method to other equations where regularity structures apply. The issue being that without localisation, it is not clear what length-scale should be used in the convolution. A Gronwall-type argument is probably necessary. A first example of such equation would be the multiplicative stochastic heat equation:

$$(\partial_t - \Delta)u = \sigma(u)\xi,$$

where ξ is the rough driving term and σ is for example a smooth Lipschitz function. This kind of equation can for example be derived from the sine-Gordon model [46],

$$(\partial_t - \Delta)u = \sin(\beta u) + \zeta,$$

where ζ is the space-time white noise in dimension 2, by expanding around the solution to the linear equation and using trigonometric identities. A scaling argument and the expansion into power series of the sine function tells us that criticality depends on the parameter β. Renormalisation for this model has been studied in the framework of regularity structures up to criticality in [20]. The regularity structure for this equation depends in particular on how well σ is approximated by its Taylor expansion. A coherence result similar to the one presented in the last chapter is believed to be also true in that case, as well as in more general cases.

One loose end of this thesis is our inability to account for exponential non-linearity in the first chapter, when it seems like the maximum principle should be even more powerful

in that case. Some results in that direction could be useful for example in the study of Liouville quantum gravity model [29], represented by the equation:

$$(\partial_t - \Delta)u = e^{\gamma u} + \xi,$$

in dimension 2. The stochastic renormalisation for this equation is similar in spirit to the renormalisation for the Sine-Gordon equation and depends on the parameter γ, but the regularity structure for this equation is not properly established up to criticality.

One more possible development of the results of this thesis is the study of initial value problems: the strong bounds obtained ensure that solutions to any such problem find themselves in a compact domain that depends only on the noise after time 1. Once the solutions are close, we may be able to use a contractivity estimate. In the case of Gaussian noise, this would provide a proof of the uniqueness of the invariant measure. This has been studied in [69] for the 2-dimensional Φ^4 model on the torus with small noise, using a similar coming down from infinity in time only. It is not clear how to remove the small noise condition, but the present thesis would at least allow to extend this result to the full space and to higher dimension.

Appendix A

Appendix:Symbolic index

In this appendix, we collect the most used symbols of the thesis, together with their meaning and the page where they were first introduced.

Symbol	Meaning	Page
ζ	Noise term	18
P_0	$(0,1) \times \{\|x\| < 1\}$	18
P_R	$(R^2,1) \times \{\|x\| < 1 - R\}$	18
$d(\cdot,\cdot)$	Parabolic distance between space-time points $z, \overline{z} \in \mathbb{R} \times \mathbb{R}^d$	26
$B(z,R)$	$\{\overline{z} = (\bar{t}, \overline{x}) \in \mathbb{R} \times \mathbb{R}^d,\, d(z,\overline{z}) < R, \bar{t} < t\}$	26
$\| \bullet \|$	L^∞ norm	26
$[\bullet]_\alpha$	Hölder seminorm of index α	26,53
Θ	Coming-down from infinity speed function	29
$\mathcal{T}$	(for Chap. 3) The set of trees needed for Φ^4_3	57
Ψ_L	Smooth compactly supported kernel, rescaled at length L	54
$(\bullet)_L$	Convolution with the kernel Ψ_L	54
Ξ	The abstract noise	104
Poly	The set $\{\mathbf{1}, \mathbf{X}_1, \dots, \mathbf{X}_d\}$	106
$\mathcal{W}$	Unplanted trees with $\|\tau\| < -2$	106
$\mathring{\mathcal{W}}$	Set of product trees in $\mathcal{W}$, namely $\mathcal{W} \setminus \{\Xi\}$	106
$\mathcal{N}$	Unplanted trees with $\|\tau\| \in [-2, 0]$, includes Poly	106
$\mathring{\mathcal{N}}$	Set of product trees in $\mathcal{N}$, namely $\mathcal{N} \setminus \mathrm{Poly}$	106

Symbol	Meaning	Page
$\mathcal{T}_r$	Unplanted trees on right-hand side of ϕ equation, $\mathcal{T}_r = \mathcal{W} \cup \mathring{\mathcal{N}}$	108
$\mathcal{T}_l$	Planted trees in expansion of ϕ, $\mathcal{T}_l = \mathcal{I}(\mathcal{T}_r) \cup \mathcal{I}(\mathrm{Poly})$	108
$\mathcal{T}$	Trees on left or right-hand side of ϕ equation, $\mathcal{T} = \mathcal{T}_l \cup \mathcal{T}_r$	108
$\widetilde{\mathcal{N}}$	$\{\tau \in \mathring{\mathcal{N}} : \lvert\tau\rvert > 1\}$	111
$\mathcal{T}_+$	$\mathcal{T} \cup \mathcal{T}^{\mathrm{rec}}$	111
$\mathcal{T}^{\mathrm{rec}}$	Planted trees that only appear for centring	112
$\mathcal{I}$	Edge of a tree corresponding to heat kernel	104
$\mathcal{I}_+^{(i)}$	Edge for derivative of heat kernel for positive renormalisation	111
$\mathcal{I}_-^{(i)}$	Edge for derivative of heat kernel for negative renormalisation	136
$[\mathbb{X}; \bullet]$	Seminorm for the local product $\mathbb{X}$ applied to a tree	122
$\mathbb{X}_\bullet$	Local product	110
$\mathbb{X}_\bullet^{\mathrm{rec}}$	Centring map	119
$\mathbb{X}_{\bullet,\bullet}$	Local path	119
ρ	cutoff function used to define local product	109
$\leqslant, \subset$	Relations on trees	116
$m(\tau)$	Number of leaves in a tree τ	107
m_Ξ	Number of noise leaves in a tree τ	107
$m_{\mathbf{x}}$	Number of $\{\mathbf{X}_i\}_{i=1}^d$ leaves in a tree τ	107
$m_{\mathbf{1}}$	Number of $\mathbf{1}$ leaves in a tree τ	107
$\lvert \bullet \rvert$	Order of a tree	107
δ	Noise is of regularity $C^{-3-\delta}$, $\delta > 0$	89
D	$(0,1) \times \{\lvert x\rvert < 1\}$	89
D_R	$(R^2, 1) \times \{\lvert x\rvert < 1 - R\}$	89
Δ	Coproduct	110
C_+	Cut map for coproduct	117
C_-	Cut map for modified coproduct	139
$\mathcal{R}$	Renormalisation operator	138
$\circ_{\mathbb{X}}$	Renormalised product of tree expansion	134
Υ	Coefficient map for solutions to equation	131

Symbol	Meaning	Page
V_γ	Expansion of the remainder solution to level γ	129
V_γ^2	Expansion of square of remainder solution to level γ	129
$V_\gamma^{((i)}$	Expansion of derivative of the remainder solution to level γ	129
U_γ^τ	Expansion of the local approximation on level τ	128

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